

# Thermal Sommerfeld effect (from lattice NRQCD)

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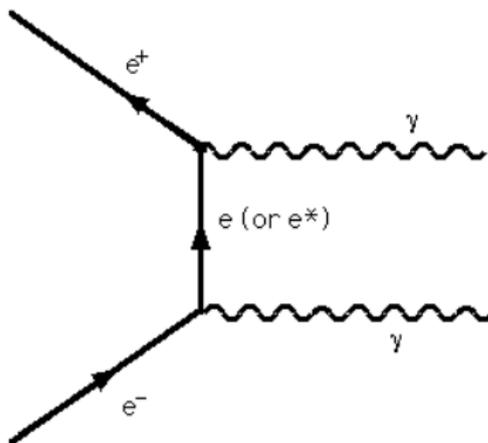
with M. Laine (ITP, U of Bern)  
(JHEP07(2016) 143, JCAP1701(2017) 013,  
and work in progress)

# Outline

- 1 Introduction
- 2 Method
- 3 Result
- 4 Discussion

# Sommerfeld Effect

- annihilation cross-section

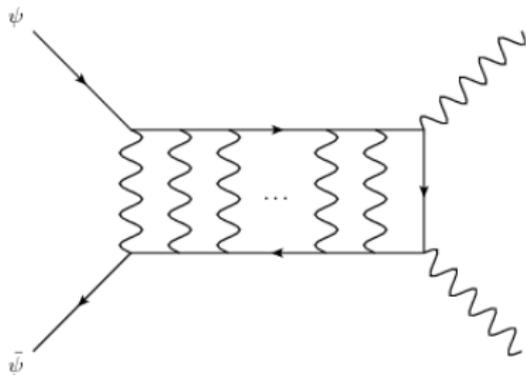


# Sommerfeld Effect

- enhancement of annihilation for slowly moving particle :

$$\sim \frac{\alpha}{v}$$

per one rung in a ladder diagram, large for small  $v$   
and need to be resummed



# Sommerfeld Effect

- Sommerfeld effect enhances the Born matrix elements

$$|\mathcal{M}_{\text{resummed}}|^2 = \mathcal{S} |\mathcal{M}_{\text{tree}}|^2$$

for the color singlet

$$S_1 = \frac{X_1}{1 - e^{-X_1}}, X_1 = C_F \frac{g^2}{4v}$$

and for the color octet

$$S_8 = \frac{X_8}{e^{X_8} - 1}, X_8 = \left(\frac{N_c}{2} - C_F\right) \frac{g^2}{4v}$$

- higher partial waves
- QCD is non-perturbative and decay of quarkonium state is described as (NRQCD factorization, G.T. Bodwin et al, PRD51 (1995) 1125)

$$\Gamma = \sum_i |\langle O_i \rangle|^2 \Gamma_{\text{partonic}}^i$$

# Thermal Sommerfeld Effect in QGP

- Lee-Weinberg equation (B.W. Lee and S. Weinberg, PRL39 (1977) 165)

$$\partial_t n = -\langle \sigma_{\text{eff}} v \rangle (n^2 - n_{\text{eq}}^2)$$

- chemical equilibration rate of heavy quark in Quark-Gluon Plasma (QGP)
- relic density of dark matter particle in early universe
- linearizing Lee-Weinberg equation (SK and M. Laine, JHEP 1607 (2016) 143)

$$\langle \sigma_{\text{eff}} v \rangle \equiv \frac{\Gamma_{\text{chem}}}{2n_{\text{eq}}}$$

# Thermal Sommerfeld Effect in QGP

- for QCD, non-perturbative definition for the chemical/kinetic equilibration rate is necessary
- equilibration rate is a real-time quantity
- lattice gauge theory is a method which can calculate non-perturbative quantities using first principles of quantum field theory
- lattice gauge theory is defined on a Euclidean space and has difficulty in calculating real-time quantity

# How to calculate: Usual way

- consider Boltzmann equation\*

$$\partial_t n \simeq -c(n^2 - n_{\text{eq}}^2) = \dot{n}_{\text{loss}} + \dot{n}_{\text{gain}}$$

with  $\dot{n}_{\text{loss}} = -cn^2$

- in equilibrium,  $n(t) = n_{\text{eq}}$  and  $\delta\dot{n} \simeq -2cn\delta n$

$$\Gamma_{\text{chem}} = \left. \frac{\delta\dot{n}}{n} \right|_{\text{eq}} = -2 \frac{\dot{n}_{\text{loss}}}{n_{\text{eq}}}$$

- then perturbatively

$$\Gamma_{\text{chem}} = \frac{2}{2N_c \int_{\mathbf{k}} f_F(E_k)} \int \int (2\pi)^4 \delta^4(P_1 + P_2 - K_1 - K_2) f_F(E_{k_1}) f_F(E_{k_2})$$

$$\left( \frac{1}{2} \sum |M_1|^2 [1 + f_B(\epsilon_{p_1})][1 + f_B(\epsilon_{p_2})] + N_f \sum |M_2|^2 [1 - f_F(\epsilon_{p_1})][1 - f_F(\epsilon_{p_2})] \right)$$

# $\Gamma_{\text{chem}}$ as a transport coefficient

- chemical equilibration as a transport coefficient (D. Bödeker, M. Laine, JHEP07 (2012) 130, 01 (2013) 037)
- treat the approach to the equilibrium as a Langevin process

$$\delta \dot{n}(t) = -\Gamma_{\text{chem}} \delta n(t) + \xi(t)$$

$$\langle\langle \xi(t) \xi(t') \rangle\rangle = \Omega_{\text{chem}} \delta(t - t'), \quad \langle\langle \xi(t) \rangle\rangle = 0$$

where  $\delta n(t)$  is the deviation from the equilibrium and  $\xi(t)$  is a stochastic noise

# $\Gamma_{\text{chem}}$ as a transport coefficient

- to access  $\Omega_{\text{chem}}$ , consider correlators (for heavy quark, or non-relativistic QCD, the heavy quark number is replaced by hamiltonian) in imaginary time

$$\Omega(\tau) = \frac{1}{V} \langle \partial_t H(\tau) \partial_t H(0) \rangle_{\text{qm}}$$

or

$$\Delta(\tau) = \frac{1}{V} \langle H(\tau) H(0) \rangle_{\text{qm}}$$

then

$$\Omega_{\text{chem}} = \Gamma_{\text{chem}} \lim_{\omega \ll \omega_{UV}} 2T \frac{\rho_{\Omega}(\omega)}{\omega} \quad \text{or} \quad = \lim_{\omega \ll T} 2T \omega \rho_{\Delta}(\omega)$$

and

$$\Gamma_{\text{chem}} = \frac{\lim_{\omega \rightarrow 0^+} 2T \frac{\rho_{\Omega}(\omega)}{\omega}}{2\chi_f M^2} \quad \text{or} \quad = \frac{\lim_{\omega \ll T} 2T \omega \rho_{\Delta}(\omega)}{2\chi_f M^2}$$

# $\Gamma_{\text{chem}}$ as a transport coefficient

- consider

$$\begin{aligned}
 & \int_{\vec{x}, \vec{y}} \langle H(\tau, \vec{x}) H(0, \vec{y}) (\psi^\dagger \chi)(\tau_1, \vec{0}) (\chi^\dagger \psi)(\tau_2, \vec{0}) \rangle_{\tau_1 > \tau_2} [\theta(\tau - \tau_1) + \theta(\tau_2 - \tau)] \\
 &= \frac{1}{Z} \text{Tr} \left[ e^{-\beta \hat{\mathcal{H}}} (\hat{\psi}^\dagger \hat{\chi})(\tau_1, \vec{0}) (\hat{\chi}^\dagger \hat{\psi})(\tau_2, \vec{0}) \int_{\vec{x}, \vec{y}} \hat{H}(0, \vec{x}) \hat{H}(0, \vec{y}) \right] \\
 &= \frac{1}{Z} \sum_{m, n} \langle q\bar{q}, m | e^{-\beta \hat{\mathcal{H}}} (\hat{\psi}^\dagger \hat{\chi})(\tau_1, \vec{0}) | n \rangle \langle n | (\hat{\chi}^\dagger \hat{\psi})(\tau_2, \vec{0}) \int_{\vec{x}, \vec{y}} \hat{H}(0, \vec{x}) \hat{H}(0, \vec{y}) | \\
 &= \frac{4M^2}{Z} \sum_{m, n} e^{-\beta E_m} e^{(\tau_1 - \tau_2)(E_m - \varepsilon_n)} \langle q\bar{q}, m | \hat{\psi}^\dagger \hat{\chi} | n \rangle \langle n | \hat{\chi}^\dagger \hat{\psi} | q\bar{q}, m \rangle
 \end{aligned}$$

$E_m$  are the states with heavy quarks,  $\varepsilon_n$  are states without heavy quarks

# $\Gamma_{\text{chem}}$ as a transport coefficient

- with  $C_{mn} = \frac{4M^2}{Z} e^{-\beta E_m} \langle q\bar{q}, m | \hat{\psi}^\dagger \hat{\chi} | n \rangle \langle n | \hat{\chi}^\dagger \hat{\psi} | q\bar{q}, m \rangle$

$$\varepsilon(\tau) = \sum_{m,n} C_{mn} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \frac{e^{(\tau_1 - \tau_2)(E_m - \varepsilon_m)}}{\tau_1 - \tau_2} [\theta(\tau - \tau_1) + \theta(\tau_2 - \tau)]$$

- then with  $\omega, \varepsilon_m \ll E_m$

$$\rho_\Delta(\omega) = \frac{\text{Im}f_1(1S_0)}{M^2} \sum_{m,n} C_{mn} \frac{e^{\beta\omega} - e^{-\beta\omega}}{\omega^2}$$

# $\Gamma_{\text{chem}}$ as a transport coefficient

- finally,

$$\begin{aligned}
 \Omega_{\text{chem}} &= \frac{4\text{Im}f_1(1S_0)}{M^2} \sum_{m,n} C_{mn} \\
 &= 16\text{Im}f_1(1S_0) \frac{1}{Z} \sum_{m,n} e^{-\beta E_m} \langle q\bar{q}, m | \hat{\psi}^\dagger \hat{\chi} | n \rangle \langle n | \hat{\chi}^\dagger \hat{\psi} | q\bar{q}, m \rangle \\
 &= 16\text{Im}f_1(1S_0) \frac{1}{Z} \text{Tr} \left[ e^{-\beta \hat{\mathcal{H}}} (\hat{\psi}^\dagger \hat{\chi})(0^+, \vec{0}) (\hat{\chi}^\dagger \hat{\psi})(0, \vec{0}) \right] \\
 &= 16\text{Im}f_1(1S_0) \langle (\psi^\dagger \chi)(0^+, \vec{0}) (\chi^\dagger \psi)(0, \vec{0}) \rangle
 \end{aligned}$$

# $\Gamma_{\text{chem}}$ as a transport coefficient

$$P_1 \equiv \frac{1}{2N_c} \text{Re} \langle G_{\alpha\alpha;ij}^\theta(\beta, \vec{0}; 0, \vec{0}) \rangle,$$

$$P_2 \equiv \frac{1}{2N_c} \langle G_{\alpha\gamma;ij}^\theta(\beta, \vec{0}; 0, \vec{0}) G_{\gamma\alpha;ji}^{\theta\dagger}(\beta, \vec{0}; 0, \vec{0}) \rangle,$$

$$P_3 \equiv \frac{1}{2N_c^2} \langle G_{\alpha\alpha;ij}^\theta(\beta, \vec{0}; 0, \vec{0}) G_{\gamma\gamma;ji}^{\theta\dagger}(\beta, \vec{0}; 0, \vec{0}) \rangle.$$

- singlet Sommerfeld factor

$$\bar{S}_1 = \frac{P_2}{P_1^2}.$$

- octet Sommerfeld factor

$$\bar{S}_8 = \frac{N_c^2 P_3 - P_2}{(N_c^2 - 1) P_1^2}.$$

# $\Gamma_{\text{chem}}$ as a transport coefficient

- P-wave Sommerfeld factor

$$\bar{S}_p = \frac{P_p}{M^2 P_1^2}$$

with

$$p_p = \text{Tr}\langle \Delta_i G_V(\beta, \vec{0}; 0, \vec{0}; i) G^\dagger(\beta, \vec{0}; 0, \vec{0}) \rangle - \text{Tr}\langle G_V(\beta, \vec{0}; 0, \vec{0}; i) \Delta_i G^\dagger(\beta, \vec{0}; 0, \vec{0}) \rangle$$

- P-wave state may have a better signal-to-noise ratio in an experimental situation or a better detection strategy
- bound states are important in thermal Sommerfeld effect and P-wave has lower melting temperature

# Lattice setup

- anisotropic Euclidean lattices (i.e., the time direction lattice spacing is different from the space direction lattice spacing,  $a_s/a_t = 3.5$ ),  
 $a_s = 0.1227(8)$  fm
- $N_f = 2 + 1$  light quark flavors ( $M_\pi \simeq 400$  MeV,  $M_K \simeq 500$  MeV)
- $24^3 \times N_t$  lattices
- $T_c = 185$  MeV,  $a_s M = 2.92$  (bottom) and 1.5 ( $\sim$  charm)
- lattices used for bottomonium at  $T \neq 0$  study (G. Aarts et al, JHEP07 (2014) 097) and electric conductivity of QGP (G. Aarts et al, JHEP-2 (2015) 186)

# Lattice setup

$$G^\theta(0, \mathbf{x}; \cdot) = \frac{\delta_{\mathbf{x}, \vec{0}}}{a_s^3},$$

$$G^\theta(a_t, \mathbf{x}; \cdot) = \left(1 - \frac{a_t \mathcal{H}_0}{2n}\right)^n U_0^\dagger(0, \mathbf{x}) \left(1 - \frac{a_t \mathcal{H}_0}{2n}\right)^n G^\theta(0, \mathbf{x}; \cdot),$$

$$G^\theta(\tau + a_t, \mathbf{x}; \cdot) = \left(1 - \frac{a_t \mathcal{H}_0}{2n}\right)^n U_0^\dagger(\tau, \mathbf{x}) \left(1 - \frac{a_t \mathcal{H}_0}{2n}\right)^n (1 - a_t \delta \mathcal{H}) G^\theta(\tau, \mathbf{x}; \cdot),$$

where  $U_0$  is a time-direction gauge link. The lowest-order Hamiltonian reads

$$\mathcal{H}_0 = -\frac{\Delta^{(2)}}{2M},$$

where  $\Delta^{(2)}$  is a discretized gauge Laplacian.

# Lattice setup

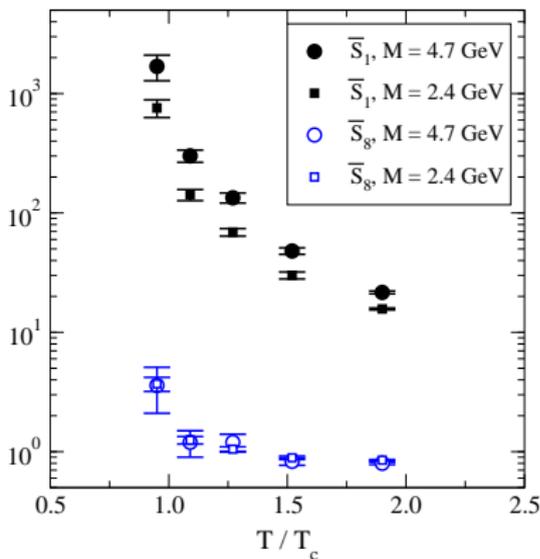
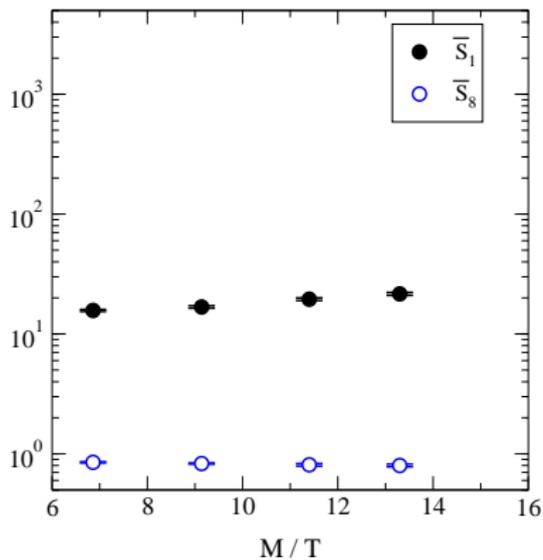
The higher order correction is

$$\delta\mathcal{H} = -\frac{(\Delta^{(2)})^2}{8M^3} + \frac{ig_0(\nabla \cdot \mathbf{E} - \mathbf{E} \cdot \nabla)}{8M^2} - \frac{g_0 \boldsymbol{\sigma} \cdot (\nabla \times \mathbf{E} - \mathbf{E} \times \nabla)}{8M^2} \\ - \frac{g_0 \boldsymbol{\sigma} \cdot \mathbf{B}}{2M} + \frac{a_s^2 \Delta^{(4)}}{24M} - \frac{a_t (\Delta^{(2)})^2}{16nM^2},$$

# Lattice result of thermal Sommerfeld factor (S-Wave)

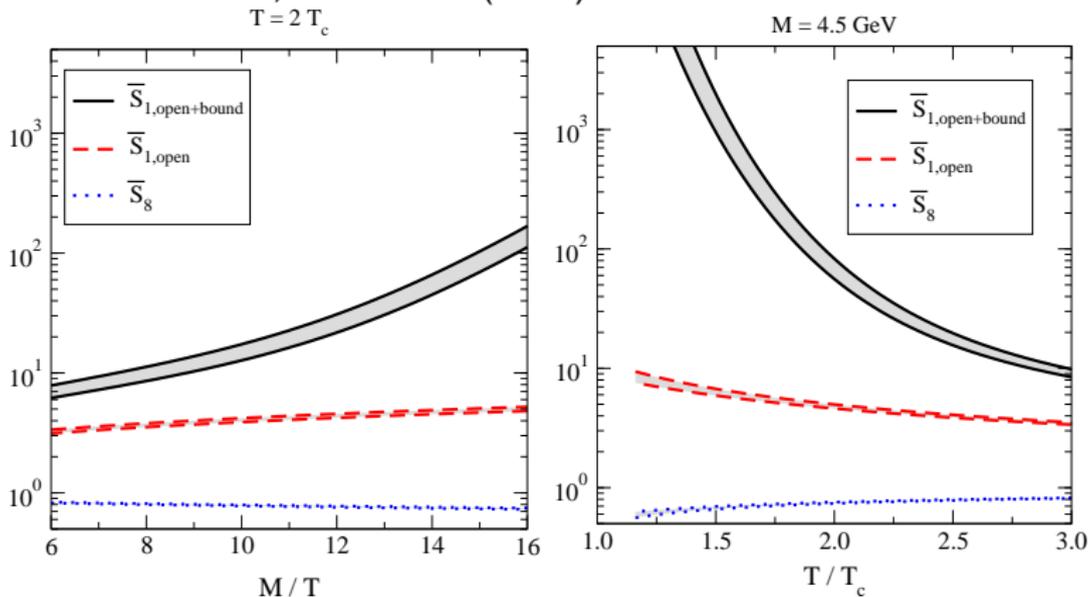
- SK and M. Laine, JHEP1607 (2016) 143

$T = 1.9 T_c$

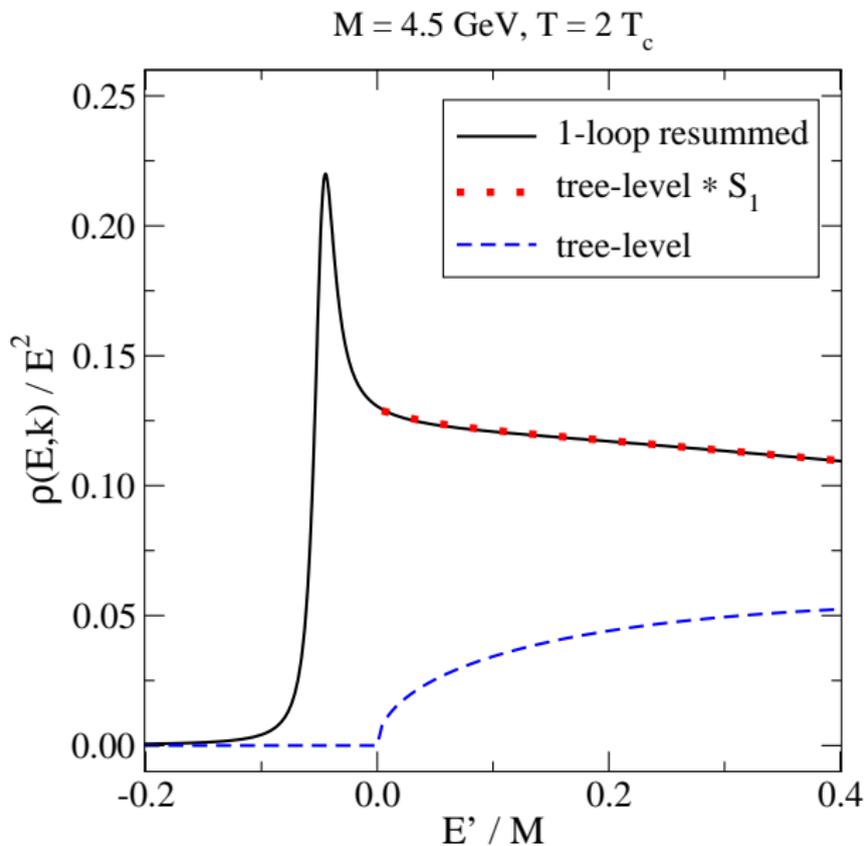


## Analytic estimate of thermal Sommerfeld factor (S-wave)

- SK and M. Laine, JHEP1607 (2016) 143



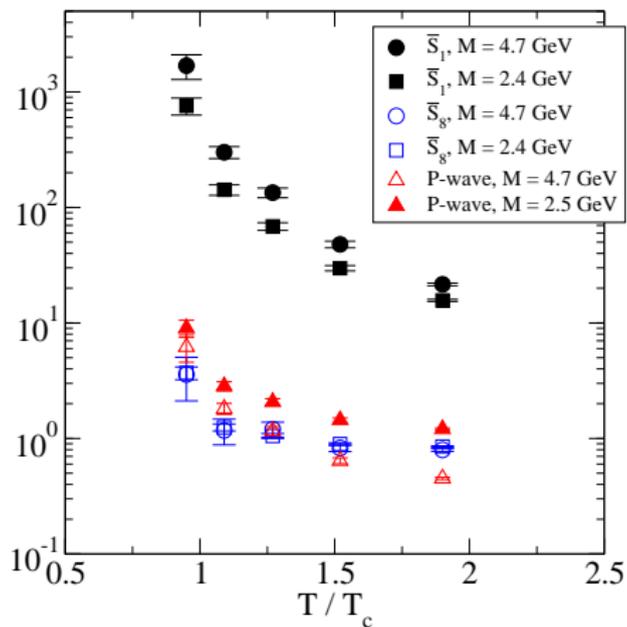
## Analytic estimate of thermal Sommerfeld factor (S-wave)



## Lattice result of thermal Sommerfeld factor (S &amp; P-Wave)

$N_t$	$T/T_c$	$a_s M$	$M/T$	$\bar{S}_1$	$\bar{S}_P$
32	0.95	2.92	26.7	1690(410)	6.2(1.6)
28	1.09	2.92	23.4	301(35)	1.8(2)
24	1.27	2.92	20.0	134(13)	1.2(1)
20	1.52	2.92	16.7	48(3)	0.64(4)
16	1.90	2.92	13.3	21.6(6)	0.45(1)
32	0.95	1.50	13.7	758(128)	9.0(1.5)
28	1.09	1.50	12.0	142(15)	2.8(3)
24	1.27	1.50	10.3	69(5)	2.1(1)
20	1.52	1.50	8.57	30(2)	1.44(6)
16	1.90	1.50	6.86	15.7(3)	1.20(2)

## Lattice result of thermal Sommerfeld factor (S &amp; P-Wave)



# Discussion

- a real time quantity, **chemical equilibration rate**, is calculated non-perturbatively using Euclidean lattice **without analytic continuation** for the first time.
- in the current relativistic heavy ion collision experiments, charm quark and bottom quarks do not chemically equilibrate.
- thermal Sommerfeld effect for S- & P-wave bottomonium annihilation is calculated using lattice NRQCD.
- **bound state** is the cause of large enhancement for S-wave thermal Sommerfeld factor. for P-wave, we expect quarkonium melting within the investigated temperature range`