# Solution of a sign problem by explicit bosonisation

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1. Loop formulation of fermionic models
   - Gross-Neveu model as a closed loop model
   - Schwinger model in the strong coupling limit
2. Simulation algorithm for vertex models
   - Worm algorithm
   - Results and efficiency
3. $QED_3$ in the strong coupling limit

# Introduction

- Simulating strongly interacting fermions continues to be a major challenge.
- E.g. Quantumchromodynamics is formally described by the Lagrange density:

$$\mathcal{L}_{QCD} = \bar{\psi}(i\not{D} - m_q)\psi - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

- This leads to the (Grand Canonical) partition function

$$Z_{GC} = \int \left(\mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\right) e^{-S_{QCD}[U;\bar{\psi},\psi]}$$

# Introduction

- One can integrate out the fermion fields to obtain the fermion determinant $\int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{-\bar{\psi}D\psi} \propto \det(D)$:

$$Z_{GC} = \int (\mathcal{D}U) \det D(U)\, e^{-S_G[U]}$$

- Problem 1: Determinant is non-local...
- Standard method is to re-express det using bosonic 'pseudo-fermions' and use Hybrid Monte Carlo (HMC):

$$\det D(U) \propto \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\, e^{-\bar{\phi}D(U)^{-1}\phi}$$

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  for $N_f$ (degenerate) fermion flavours.

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# Introduction

- Problem 2: Critical slowing down towards the chiral limit . . .
  - fermions become massless,
  - correlation length of the fermionic 2-point function diverges,
  - Dirac operator $D(U)$ develops very small modes,
  - the inverse $D(U)^{-1}$ becomes ill-conditioned.
- Problem 3: Possible phase of $\det D$ forbids MC simulations

  $\Rightarrow$ Sign Problem

  - e.g. $N_f = 2N + 1$ Wilson fermions can not be simulated.

## Introduction

- We propose a novel approach circumventing these problems:
  - based on (high-temperature) expansion of the fermion action,
  - reformulates fermionic systems as statistical closed loop models, i.e. $q$-state vertex models,
  - eliminates critical slowing down,
  - allows simulations directly in the massless limit.
- Applicable to the
  - Gross-Neveu model in $d = 2$ dimensions,
  - Schwinger model in the strong coupling limit in $d = 2$ and 3.

$\Rightarrow$ solution of the sign problem

## Definition of the model

- Euclidean lagrangian density in 2D [Gross, Neveu '74]

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \partial\!\!\!/ \, \psi^\alpha(x) - \frac{g^2}{2} \left( \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \psi^\alpha(x) \right)^2 ,$$

  where $\psi^\alpha(x)$ are 2-component Dirac spinors and $\alpha$ flavour index.

- Introduce a scalar field $\sigma(x)$ conjugate to $\sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x)\psi^\alpha(x)$:

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \partial\!\!\!/ \, \psi^\alpha(x) + \frac{1}{2g^2}\sigma(x)^2 + \sigma(x) \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x)\psi^\alpha(x).$$

## Properties

- The Gross-Neveu model
  - is renormalisable and asymptotically free,

$$\beta(g) = -\frac{N-1}{2\pi} g^3 + O(g^5),$$

  - has a $O(2N) \times \Gamma$-symmetry where $\Gamma$ is the discrete chiral symmetry

$$\Gamma: \qquad \psi \to \gamma_5 \psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5, \quad \sigma \to -\sigma,$$

  - exhibits spontaneous breaking of the discrete chiral symmetry

    $\Rightarrow$ fermions acquire non-vanishing mass $\sigma_0 = \langle \sigma \rangle$ (dimensional transmutation).

Note: there is no Goldstone boson due to $\Gamma$ being a discrete symmetry.

## Large-$N$ limit

- In the large-$N$ limit with $\lambda = g^2 N$ fixed, the model can be solved analytically:
  - Integrate out the fermions to obtain $Z = \int_{[d\sigma]} \exp\{-S_{\text{eff}}\}$,

$$S_{\text{eff}} = N \left\{ \int_{[dx]} \frac{\sigma(x)^2}{2\lambda} - \operatorname{Tr}\log\left[\partial\!\!\!/ + \sigma\right] \right\}.$$

  - The minimum of the effective potential is given by

$$\partial_{\sigma(x)} S_{\text{eff}}/N = \frac{\sigma(x)}{\lambda} - \partial_{\sigma(x)} \operatorname{Tr}\log\left[\partial\!\!\!/ + \sigma\right] = 0, \ \forall x.$$

## Spectrum of the GN model

- To leading order in $1/N$ the spectrum consists of [Dashen, Hasslacher, Neveu '75; Feinberg, Zee '97]

$$m_1 = \sigma_0 \sim \Lambda \exp\left\{-\frac{\pi}{\lambda}\right\}, \quad \text{single fermion,}$$

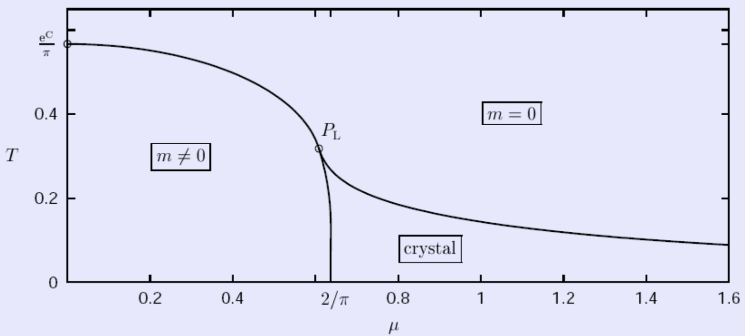
$$m_n = \sigma_0 \cdot \frac{2N}{\pi} \sin\left(\frac{n\pi}{2N}\right), \quad \text{n-fermion bound state,}$$

$$m_B = \sigma_0 \cdot \frac{2N}{\pi}, \quad \text{kink-antikink state ('baryon').}$$

- The GN model possesses a rich $\mu$-$T$ phase structure [Dashen, Ma, Rajaraman '75; Wolff '85; Karsch, Kogut, Wyld '87].

## The revised phase diagram

- The structure of cold baryonic matter has only recently been clarified [Thies, Schön, Brzoska, Schnetz, Urlichs '00-'06; Dunne, Baym '08].
- In addition to the massive and massless Fermi gas, there is a new baryonic crystal phase at low temperature:

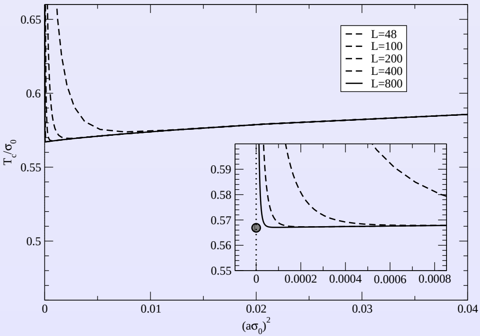



- $\mu_c = \frac{2}{\pi}$ consistent with $m_B$, no first order transition at $\mu \neq 0$.

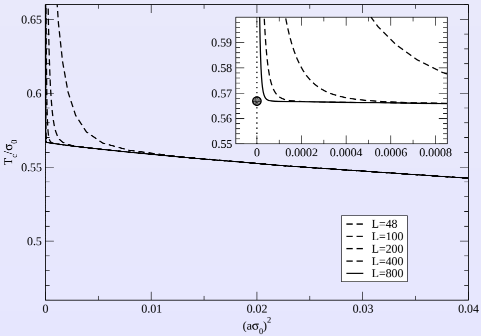
# GN model on the lattice

- One can use staggered or overlap fermions, both of which preserve the discrete chiral symmetry [de Forcrand, Wenger '06].
- Scaling of $T_c/\sigma_0$ vs $(a\sigma_0)^2$ $\Rightarrow$ universality at work:

staggered operator




overlap operator



## GN model with Majorana fermions

- Most natural formulation in terms of Majorana fermions.
- For the Wilson lattice discretisation

$$\mathcal{L} = \frac{1}{2}\xi^T \mathcal{C}(\gamma_\mu \tilde{\partial}_\mu - \frac{1}{2}\partial^* \partial + m)\xi - \frac{g^2}{4}\left(\xi^T \mathcal{C}\xi\right)^2$$

  - $\xi$ is a real, 2-component Grassmann field,
  - $\mathcal{C} = -\mathcal{C}^T$ is the charge conjugation matrix.
- The discrete chiral symmetry $\xi \to \gamma_5 \xi$ is broken explicitly:
  $\Rightarrow$ restored in the continuum by fine tuning $m \to m_c$.

# GN model with Majorana fermions

- Each pair of Majorana fermions may be considered as one Dirac fermion:

$$\psi = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\xi_1^T - i\xi_2^T)\mathcal{C}.$$

  making the $O(2N)$ flavour symmetry explicit.

- At $g = 0$, integrating the fermions yields the Pfaffian

$$Z_{\mathrm{GN}} = \mathrm{Pf}\left[\mathcal{C}(\gamma_\mu \tilde{\partial}_\mu + m - \frac{1}{2}\partial^* \partial)\right]^{2N}.$$

- $2N$ Majorana fermions $\equiv N$ Dirac fermions with

$$(\mathrm{Pf} D)^{2N} = (\det D)^N .$$

# GN model with Majorana fermions

- At $g \neq 0$ we introduce a scalar field $\sigma \propto \xi^T \mathcal{C} \xi$ as before:

$$S = \frac{1}{2} \sum_x \xi^T(x) \mathcal{C} (2 + m + \sigma(x)) \xi(x) - \sum_{x,\mu} \xi^T(x) \mathcal{C} \frac{1-\gamma_\mu}{2} \xi(x+\hat{\mu}).$$

- Using the nilpotency of Grassmann elements we expand the Boltzmann factor

$$\int \mathcal{D}\xi \prod_x \left(1 - \varphi(x) \xi^T(x) \mathcal{C} \xi(x)\right) \prod_{x,\mu} \left(1 + \xi^T(x) \mathcal{C} P(\mu) \xi(x+\hat{\mu})\right)$$

where $\varphi(x) = 2 + m + \sigma(x)$ and $P(\pm\mu) = \frac{1}{2}(1 \mp \gamma_\mu)$.

# GN model with Majorana fermions

- At each site, the fields $\xi^T \mathcal{C}$ and $\xi$ must be exactly paired to give a contribution to the path integral:

$$\int \mathcal{D}\xi \prod_x \left(\varphi(x)\xi^T(x)\mathcal{C}\xi(x)\right)^{m(x)} \prod_{x,\mu} \left(\xi^T(x)\mathcal{C}P(\mu)\xi(x+\hat{\mu})\right)^{b_\mu(x)}$$

  with occupation numbers

  - $m(x) = 0, 1$ for monomers,
  - $b_\mu(x) = 0, 1$ for bonds (or dimers),

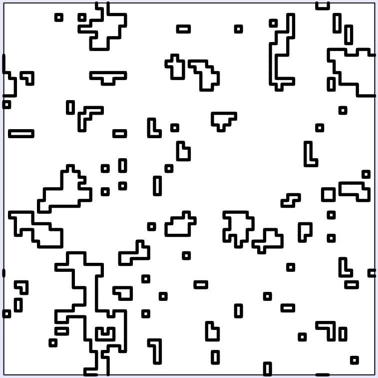
  satisfying the constraint

$$m(x) + \frac{1}{2}\sum_\mu b_\mu(x) = 1.$$

# GN model as a closed loop model

- Only closed, non-intersecting paths survive the integration

  $\Rightarrow$ closed loop representation in terms of monomers and dimers.



# GN model as a closed loop model

- Each empty site carries the monomer weight $\varphi(x)$.
- Loops are non-oriented due to Majorana characteristics

$$\xi^T(x)\mathcal{C}P(\mu)\xi(x+\hat{\mu}) = \xi^T(x+\hat{\mu})\mathcal{C}P(-\mu)\xi(x)$$

- The weight $\omega$ of each loop $\ell$ is given by the Dirac structure

$$\text{Tr}[P(\mu_1)P(\mu_2)\ldots P(\mu_n)] = \text{Tr}[P(-\mu_n)\ldots P(-\mu_2)P(-\mu_1)] \in \mathbb{Z}_2$$

and can be calculated analytically [Stamatescu '82; Wolff '07]:

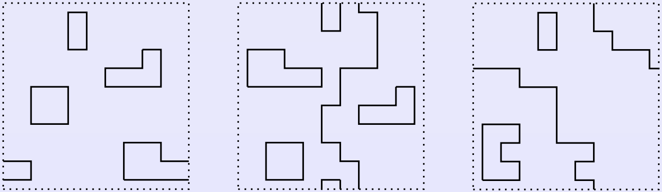
$$|\omega(\ell)| = \left(\frac{1}{\sqrt{2}}\right)^c, \quad c = \text{number of corners}$$

- The phase of $\omega(\ell)$ depends on the geometrical shape of $\ell$

$\Rightarrow$ no probabilistic interpretation in general.

# Boundary conditions

- For $d = 2$, $\arg[\omega(\ell)] = 0$ unless the loop winds around a boundary.
- In that case, $\arg[\omega(\ell)]$ depends on the fermionic boundary conditions being periodic or anti-periodic:



$\mathcal{L}_{00}$ $\mathcal{L}_{01}$ $\mathcal{L}_{11}$

$\Rightarrow$ classify all configurations into equivalence classes $\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}$.

# Boundary conditions

- Partition function summing over all non-oriented, self-avoiding loops

$$Z = \sum_{\{\ell\}\in\mathcal{L}} \omega[\ell] \prod_{x\notin\ell} \varphi(x), \quad \mathcal{L} \in \mathcal{L}_{00} \cup \mathcal{L}_{10} \cup \mathcal{L}_{01} \cup \mathcal{L}_{11}$$

  represents a system with unspecified fermionic b.c.

- For fixed fermionic boundary conditions we have

$$\begin{aligned}
Z_\xi^{00} &= Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} - Z_{\mathcal{L}_{11}} \\
Z_\xi^{10} &= Z_{\mathcal{L}_{00}} + Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} + Z_{\mathcal{L}_{11}} \\
Z_\xi^{01} &= Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} + Z_{\mathcal{L}_{01}} + Z_{\mathcal{L}_{11}} \\
Z_\xi^{11} &= Z_{\mathcal{L}_{00}} + Z_{\mathcal{L}_{10}} + Z_{\mathcal{L}_{01}} - Z_{\mathcal{L}_{11}}
\end{aligned}$$

# Determination of the critical mass

- $Z_\xi^{00}(m)$ has a zero mode at $m = m_c = 0$.
- Use this as the criterion for the determination of $m_c$

$$Z_\xi^{00}[m = m_c] = 0$$

- This can be extended to the interacting GN model with $N$ flavours:
  - for any given background $\sigma(x)$ one can show that

$$Z_\xi^{00}[\sigma(x)] = -Z_\xi^{00}[-\sigma(x)],$$

  - choose any odd number of fermions to have periodic b.c.

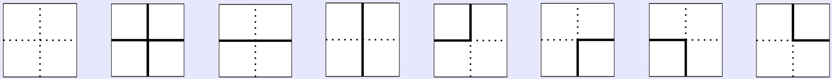
$$Z_{\text{GN}}[m = m_c] = 0.$$

# Equivalence to the 8-vertex model

- The loop formulation of the GN model is equivalent to a special case of the 8-vertex model [Lieb '67; Sutherland '70; Fan, Wu '70]:

$$Z_{8-vertex} = \sum_{\ell \in \mathcal{L}} \prod_x \omega(x).$$

- Generically we have eight vertices with weights:



$\omega_1$ $\omega_2$ $\omega_3$ $\omega_4$ $\omega_5$ $\omega_6$ $\omega_7$ $\omega_8$

- For the GN model we have [Scharnhorst '96; Gattringer '98; Wolff '07]:

$$\begin{aligned} \omega_1 &= \varphi(x), & \omega_3 = \omega_4 &= 1, \\ \omega_2 &= 0, & \omega_5 = \omega_6 = \omega_7 = \omega_8 &= \frac{1}{\sqrt{2}}. \end{aligned}$$

# 8-vertex model

- The 8-vertex model can be solved analytically in two particular cases [Fan, Wu '70; Baxter '71; Samuel '80]:
  - under the free fermion condition where

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8.$$

  - in the 'zero field' limit where

$$\begin{aligned} \omega_1 &= \omega_2, & \omega_3 &= \omega_4, \\ \omega_5 &= \omega_6, & \omega_7 &= \omega_8, \end{aligned}$$

$\Rightarrow$ The free Majorana GN model is critical at $m = 0$ and $-2$.

# Examples of 8-vertex models

- Ising model in low- and high-temperature expansion,
- Ising model with next-to-nearest and four-spin coupling,

  [Wu '71; Kadanoff, Wegner '71; Jüngling '75]

- GN model with Majorana Wilson fermions,
- GN model with $N_f$ Dirac Wilson fermions.

  [Scharnhorst '96; Gattringer '98; Wolff '07]

- $\text{QED}_2$ at $\beta = 0$ with Wilson fermions,

  [Salmhofer '92]

# Interacting 8-vertex models

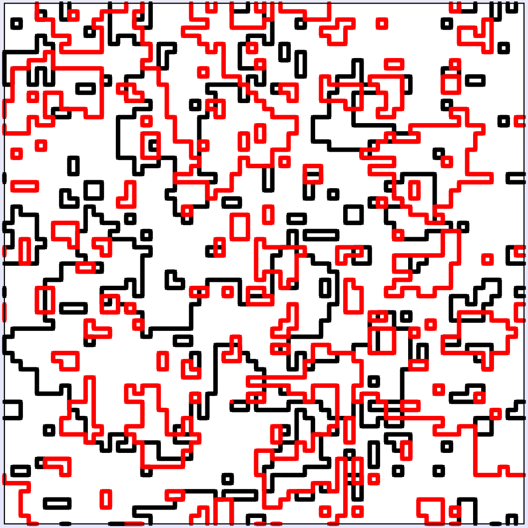
- For a single Majorana Wilson fermion, the term $\propto g$ is irrelevant:
  - weights are factorised into terms at each $x$,
  - integration over $\sigma(x)$ can be done, yielding $m+2$.
- For $N > 1$, interaction couples the $N$ flavours, but $\omega_1$ can still be calculated analytically:
  - monomer weight only depends on the occupation number

$$\omega_1(n) = \frac{1}{\sqrt{2\pi g^2}} \int d\sigma e^{-1/(2g^2)\sigma^2} (m+2+\sigma)^n$$

$\Rightarrow N$ coupled 8-vertex models

- Generic case $\omega_1(x) = \varphi(x)$ involves no complication.

# $N_{\text{Dirac}} = 1$ GN model aka Thirring model



## Schwinger model

- The Schwinger model on the lattice is defined by

$$Z_{\mathrm{SM}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}U \quad e^{-S_F^W[\bar{\psi},\psi,U]-\beta S_G[U]}$$

where $\beta = \frac{1}{g^2 a^2}$ and the gauge field $U = e^{i\phi} \in U(1)$.

- The phase diagram of the model is expected to have a critical line from $\beta = 0, m_c(0)$ to $\beta = \infty, m_c(\infty) = 1/(2d)$.
- Properties of the model include
  - confinement, leading to chargeless particles,
  - fermion condensation due to $U(1)$ axial anomaly.

# Schwinger model

- Deconstruct the Dirac fermion into a pair of Majorana fermions $\xi^{(1)}, \xi^{(2)}$.
- Hopping terms contain $U(1)$ phase from gauge field $\phi_\mu(x)$:

$$\sum_{x,\mu} \xi^T(x)\, \mathcal{C}\, e^{\pm i\phi_\mu(x)} P(\pm\mu)\xi(x \pm \hat{\mu})$$

- Bonds now carry additional factor $\propto \cos(\phi_\mu(x))$.
- Gauge field also introduces interaction between the two Majorana flavours:

$$\xi^{(1)} \to \xi^{(2)} \;\propto \sin(\phi_\mu(x)), \quad \xi^{(2)} \to \xi^{(1)} \;\propto -\sin(\phi_\mu(x))$$

- One can show that in the strong coupling limit all phases are cancelled

$\Rightarrow$ explicit bosonisation

# Schwinger model in the strong coupling limit

- Consider a singly occupied bond with weight $\propto \sin(\phi_\mu(x))$ or $\cos(\phi_\mu(x))$:

$$\frac{1}{2\pi}\int d\phi \cos\phi = 0, \quad \frac{1}{2\pi}\int d\phi \sin\phi = 0,$$

- Only doubly occupied bonds survive gauge field integration:

$$\frac{1}{2\pi}\int d\phi \cos^2\phi = \frac{1}{2}, \quad \frac{1}{2\pi}\int d\phi \sin^2\phi = \frac{1}{2},$$

- The two Majorana fermions are tightly bound together.

## Schwinger model in the strong coupling limit

- Each bond carries the weight $\frac{1}{2} + \frac{1}{2} = 1$ since the two combinations can not be distinguished.
- The signs from the fermionic loops and the Dirac traces are 'squared away',

$$(-1)_f^2 \, \mathrm{Tr}[P(\mu_1) \ldots P(\mu_n)]^2 = 1.$$

- Corner weights are squared as well as the monomer weights:

$$\begin{aligned} \omega_1 &= (m+2)^2, & \omega_3 = \omega_4 &= 1, \\ \omega_2 &= 0, & \omega_5 = \omega_6 = \omega_7 = \omega_8 &= \tfrac{1}{2}, \end{aligned}$$

$\Rightarrow$ all weights are positive

# Schwinger model in the strong coupling limit

- Loops do not distinguish periodic or anti-periodic boundary conditions, so

$$Z_{\text{SCSM}} = Z_{\mathcal{L}_{00}} + Z_{\mathcal{L}_{10}} + Z_{\mathcal{L}_{01}} + Z_{\mathcal{L}_{11}}.$$

- The combination $Z^{\mathcal{CC}}_{\text{SCSM}} \equiv Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} - Z_{\mathcal{L}_{11}}$ describes $\mathcal{C}$-periodic b.c.

  $\Rightarrow Z^{\mathcal{CC}}_{\text{SCSM}}(m = m_c) = 0$ defines the critical point.

- Bosonisation directly with Dirac fermions is perhaps more intuitive:
  - oriented loops pair up in oppositely oriented directions,
  - chargeless boson $\equiv$ particle and anti-particle,
- For $N_f \geq 2$ we can also introduce a chemical potential.

## Simulation algorithms

- *Problem*: local loop updates involving plaquette moves
  - can not change between $\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}$,
  - are highly inefficient.

- *Solution 1*: imbed the bonds of the loops in a Ising system,

  $\Rightarrow$ efficient cluster algorithms can be constructed [Wolff '07].

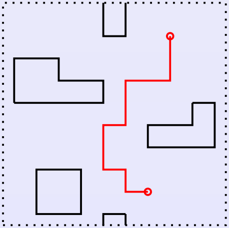
- *Solution 2*: enlarge the configuration space by one open string [Prokof'ev, Svistunov '01]:

  $\Rightarrow$ worm algorithm.

- Both solutions essentially eliminate critical slowing down.

# Particle insertions

- The open string corresponds to the insertion of a Majorana fermion pair $\{\xi^T(x)\mathcal{C}, \xi(y)\}$ at position $x$ and $y$:



$\Rightarrow$ open string samples directly the correlation function

# Two-point functions from open strings

- In the fermionic models the open string corresponds

$$G(x,y) \propto \int \mathcal{D}\xi e^{-S_{\mathrm{GN}}} \xi(x)\xi(y)^T \mathcal{C}$$

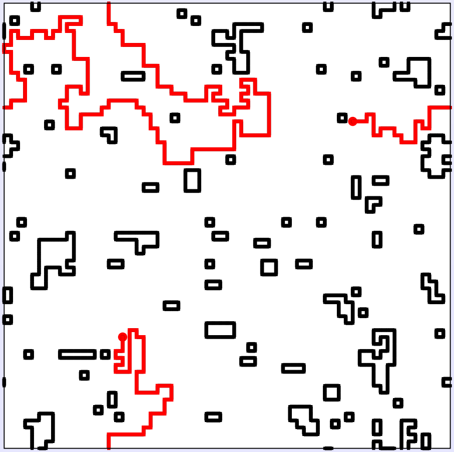
and

$$G(x,y) \propto \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_{\mathrm{SCSM}}} \psi(x)\bar{\psi}(x)\, \psi(y)\bar{\psi}(y)$$

- This is the reason why critical slowing down is eliminated:
  - at critical point correlation length $\zeta$ diverges,
  - configurations are updated on length scales up to $O(\zeta)$.

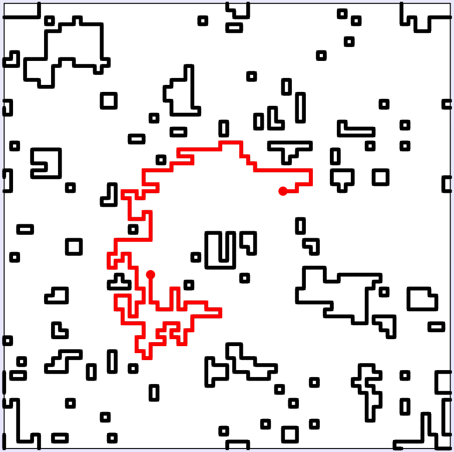
# Two-point functions from open strings

- $N = 1$ Majorana GN model, slightly above criticality:



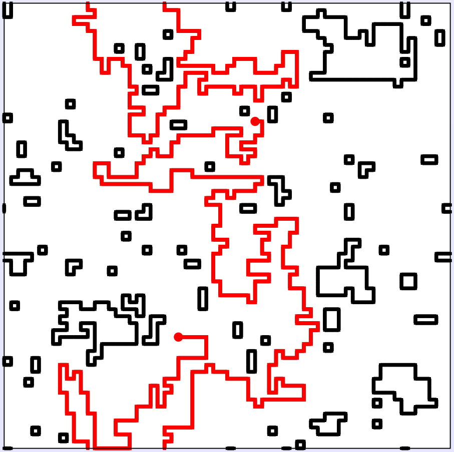
# Two-point functions from open strings

- $N = 1$ Majorana GN model, exactly at criticality:



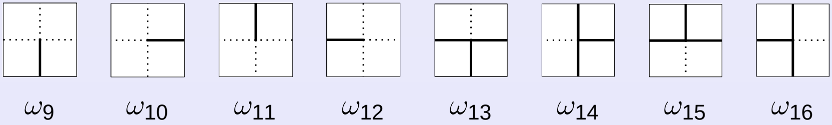
# Two-point functions from open strings

- $N = 1$ Majorana GN model, slightly below criticality:



# Worm algorithm for vertex models

- In the vertex language we introduce open tiles:




  with weights $\omega_9 - \omega_{16}$ according to the correlation function.

- For fermionic models $\omega_{13} - \omega_{16}$ are explicitely forbidden by Pauli's principle, but

  $\Rightarrow$ we include them anyway for efficiency.

# Worm algorithm

- Head and tail of the worm can move around.
- Local moves $x \to y$ with $\{v_x, v_y\} \to \{v'_x, v'_y\}$ are determined by Metropolis

$$P(x \to y) = \min\left[1, \frac{\omega_{v'_x}}{\omega_{v_x}} \frac{\omega_{v'_y}}{\omega_{v_y}}\right] .$$

- Contact with partition function $Z$ each time open string closes:
  $\Rightarrow$ global update results from sequence of local moves.
- Inbetween, the open string samples directly the 2pt. function:

$$\langle G(x,y)\rangle_Z = G(x,y)/Z .$$

- Keep track of Dirac structure by adding up $\prod_{\mu\in\ell} P(\mu)$.

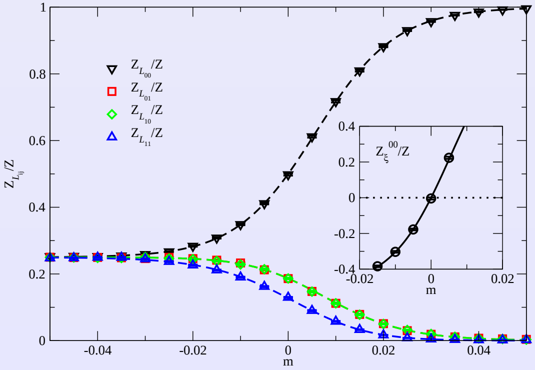
# Worm algorithm

- Worm can be made to break and reconnect existing loops.
- Worm can arbitrarily wind around the lattice

  $\Rightarrow$ all sectors $\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}$ are visited.
- Similar ideas have been around for a long time [Thun et al. '8?; Evertz, Lana, Marcu '93; Prokof'ev, Svistunov '01, Adams, Chandrasekharan '03].
- Algorithm is applicable to any vertex model in arbitrary dimensions $d$.

# Consistency check

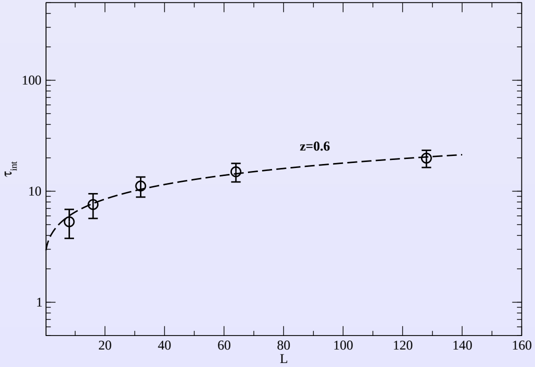
- Use solvable $N = 1$ Majorana GN model as test ground.
- Ratios $Z_{\mathcal{L}_{ij}}/Z$ on a $128^2$ lattice:




$\Rightarrow Z_\xi^{00} = Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} - Z_{\mathcal{L}_{11}}$ has zero mode at $m_c = 0$.

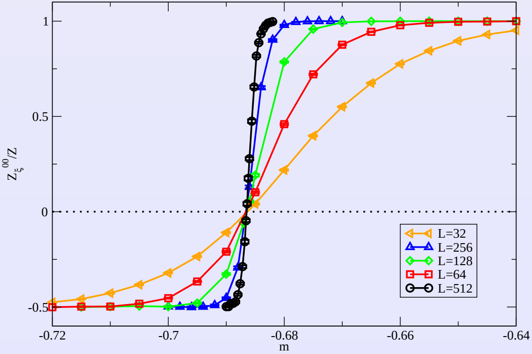
# Autocorrelation times for worm agorithm

- Measure $\langle \xi^T \mathcal{C} \xi \rangle_{Z_\xi}$ for which $\langle K(x) \rangle_{loop}$ (monomer density) is an improved estimator.
- Elimination of critical slowing down at $m = 0$:



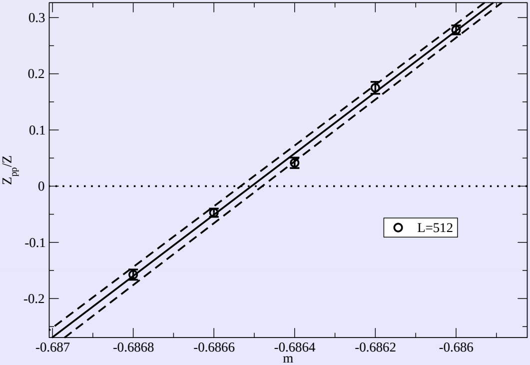
# Schwinger model at strong coupling

- Schwinger model at $g = \infty$ as a non-trivial example.
- Ratios $Z_{\mathcal{L}_{ij}}/Z$ on various lattices:



# Schwinger model at strong coupling

- Use $Z_\xi^{00}(m_c) = 0$ as a definition for $m_c$:

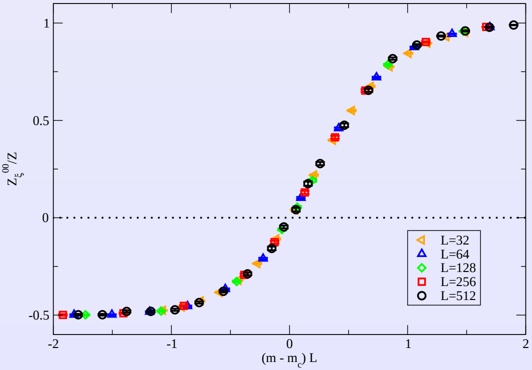



$\Rightarrow m_c = -0.686506(27)$

consistent with $m_c = -0.6859(4)$ [Gausterer, Lang '95].

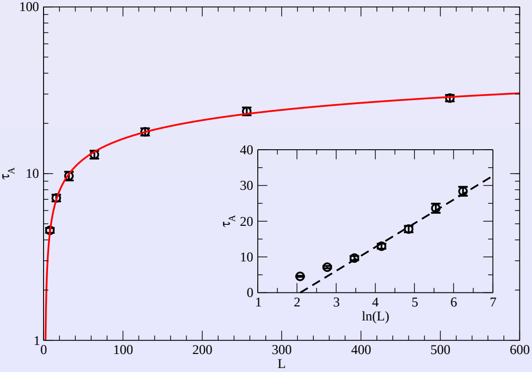
# Schwinger model at strong coupling

- Calculations indicate 2nd order phase transition:
  $\Rightarrow$ universality class of the Ising model (with $\nu = 1$)
- Apply corresponding finite size scaling $(m - m_c)L^{\nu}$:



# Schwinger model at strong coupling

- Efficiency of the algorithm at $m = m_c$:




$\Rightarrow$ $\tau_A \propto L^z$ with $z = 0.25(2)$,
alternatively, $\tau_A = -13.8(1.9) + 6.6(4)\ln(L)$

## Extension to higher dimensions

- What about extending this to higher dimensions $d > 2$?
- Hopping expansion to all orders is not restricted to any dimension. . .
- Problem: in $d = 3$ phases of the loops $\ell$ become complex,

$$\mathrm{Tr}\left[\prod_{\mu\in\ell} P(\mu)\right] \in \mathbb{Z}_8 .$$

- Nevertheless, bosonisation works for $\mathrm{QED}_3$ in the strong coupling limit in a very peculiar way.

## QED in $d = 3$ and parity anomaly

- Consider QED in $d = 3$, i.e. a massless Dirac fermion interacting with a massless Abelian gauge field.
- Properties of the theory include
  - super-renormalisability, gauge coupling $ag^2$ dimensionless,
  - confinement, asymptotic freedom,
  - no chiral symmetry, but instead a parity anomaly...

# QED in $d = 3$ and parity anomaly

- The theory admits a gauge-invariant, but parity-violating mass term for the photon $\propto \varepsilon_{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma$:
  - topological Chern-Simons action term,
  - photon mass is quantised.
- Mass term $\propto \bar{\psi}\psi$ for the two-component Dirac fermion also violates parity.
- A mass for one particle induces a mass for the other perturbatively.
- No parity-breaking mass terms are generated from $\mathcal{P}$-invariant theory to all orders in perturbation theory,

  $\Rightarrow$ non-perturbative spontaneous parity breaking

## QED in $d = 3$ and parity anomaly

- Solution: consider fermion doublet, i.e. 4-component spinor $\psi$ with fermion mass

$$\bar{\psi}\psi = m\bar{\psi}_1\psi_1 - m\bar{\psi}_2\psi_2.$$

- Mass term is invariant under $\psi_1 \to \sigma_2\psi_2, \psi_2 \to \sigma_2\psi_1$.
- Essentially this corresponds to defining $\gamma$-matrices

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix},$$

and

$$\gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

anticommuting with $\gamma_1, \gamma_2, \gamma_3$ and defining a chiral symmetry.

## QED in $d = 3$ and parity anomaly

- Wilson fermions break parity explicitely through Wilson term [So '85; Coste, Lüscher '89],

  $\Rightarrow$ in analogy to chiral symetry in 4D.

- Resulting theory is non-universal:
  - coefficient of induced CS-term depends on $r$,
  - for $r \neq 1$ deconfinement,

- In the strong coupling limit CS-term survives. . .

- Consider now fermion doublet, i.e. 4-component spinor with the two independent $\gamma$-representations

  $\Rightarrow$ avoid parity anomaly, study S$\chi$SB at $m = m_c$

## QED in $d = 3$ and parity anomaly

- This corresponds to a pair of fermions with opposite sign for mass and Wilson term

$$(m, r) \quad \text{and} \quad (-m, -r)$$

- Break it down to a pair of Majorana fermions as before.
- Fermions hop with $P(\mu) \leftrightarrow P(-\mu)$ interchanged:

$$\text{Tr}\,[P(\mu_1)P(\mu_2)\ldots P(\mu_n)] = \text{Tr}\,[P(-\mu_1)P(-\mu_2)\ldots P(-\mu_n)]^*$$

- At $\beta = 0$, Dirac phases combine to $\text{Tr}\,[\Gamma]\,\text{Tr}\,[\Gamma]^* = +1$

$\Rightarrow$ sign problem is avoided by bosonisation

# Conclusions

- We have applied the Worm algorithm to fermionic and generic vertex models.
- It relies on sampling directly the 2-point correlation function.
- It essentially eliminates critical slowing down.
- Opens the way to simulate
  - generic vertex models in arbitrary dimensions,
  - GN model with any number of flavours,
  - Thirring model,
  - Schwinger model at strong coupling in $d = 2$ and 3,
  - fermionic models with Yukawa-type interactions,

  all with Wilson fermions.