What can we do with the complex action?

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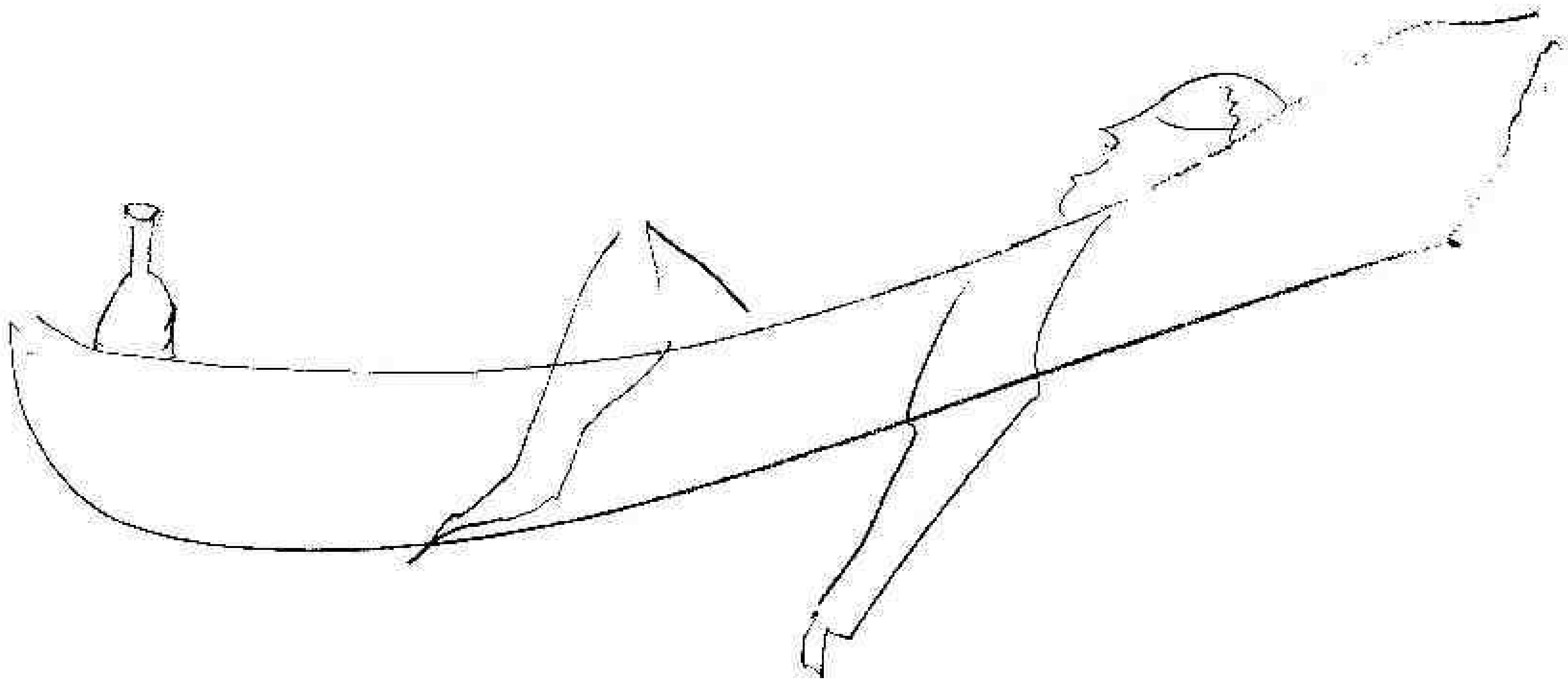


Figure 1: Random Walk.

I Generals

1. Complex action problems.
2. Real time non-equilibrium dynamics.
3. Non-zero chemical potential.
4. Some approaches and methods of solution.

II Specials

5. Reweighting applied to the chemical potential problem.
6. Stochastic processes for complex action.
7. General remarks.

1. Complex action in Quantum Field Theory

Starting from a real action problem and continuing the parameters in the complex plane

One must take into account the analyticity properties of the partition function and expectation values. For simple models these can be found out, which allows to observe non-trivial convergence properties, etc. These can be related to phase transitions, to solution multiplicity of Schwinger Dyson equations, to the behaviour of complex Langevin simulation, etc. (*many papers since the 1980's: C. Bender, Gausterer, Guralnik, Salcedo, Namiki, Schültke, Nakazano, ...* -)

Starting from special problems with complex action

- real time (Minkowski) path integral,

- non-zero chemical potential in euclidean LGT

- $\theta-$ vacua

- ...

2. Minkowski path integral

- equilibrium parameters (masses, decay rates) can be obtained from euclidean simulations (lattice theory)

- time evolution, especially non-equilibrium field theory: perturbative methodes, classical dynamics, etc
  (*Berges, Smit, Bödeker, Garcia-Perez, ...*)

Nonequilibrium dynamics, generating functional:

$$
\begin{aligned}
Z[J;\rho] &= \mathrm{Tr}\left\{\rho\, T_{\mathcal{C}}\, e^{i\int_{\mathcal{C}} J(x)\Phi(x)}\right\} \\
&= \int \mathrm{d}\varphi_1 \mathrm{d}\varphi_2\, \rho(\varphi_1,\varphi_2) \int\limits_{\varphi_1}^{\varphi_2} D'\varphi\, e^{i\int_{\mathcal{C}}(L(x)+J(x)\varphi(x))}\,. \qquad (1)
\end{aligned}
$$

with initial-time density matrix $\rho(\varphi_1,\varphi_2)$ and $\mathcal{C}$ a closed contour extending along the real t axis.

Expectation of a real-time observable:

$$
\begin{aligned}
\langle \mathcal{A}(\varphi)\rangle &= \int \mathrm{d}\varphi_1 \mathrm{d}\varphi_2\, \rho(\varphi_1,\varphi_2) \int_{\varphi_+(0)=\varphi_1}^{\varphi_-(0)=\varphi_2} D'\varphi_- D'\varphi_+ e^{iS[\varphi_+]-iS[\varphi_-]}\mathcal{A}(\\
&= \int D\varphi_- D\varphi_+ e^{iS_\rho[\varphi_+,\varphi_-]}\mathcal{A}(\varphi_+)\,.
\end{aligned}
$$

3. Non-zero chemical potential.

- QCD
- Heavy Dense Matter model
- Various approximations using the fact that the chemical potential selects the Polyakov loops in the fermionic determinant: Polyakov loop models, Potts model, etc(*Green and Karsch 1984, Karsch and Wilde 1985, Karsch and Stickan 2000, Alford et al, 2001,...*)
- Relativistic Bose gas (*Aarts 2008*)
- ...

QCD grand canonical partition function (Wilson fermions):

$$Z = \int DU \, \mathrm{e}^{-S}, \quad S = S_{YM} - \log \det W \tag{3}$$

$$\begin{aligned} W &= 1 - \kappa \sum_{i=1}^{3} \left(\Gamma_{+i} U_{x,i} T_i + \Gamma_{-i} U^{\dagger}_{x,i} T_{-i} \right) \\ &\quad -\kappa \left(\mathrm{e}^{\mu} \Gamma_{+4} U_{x,4} T_4 + \mathrm{e}^{-\mu} \Gamma_{-4} U^{\dagger}_{x,4} T_{-4} \right) \end{aligned} \tag{4}$$

Here T are lattice translations, $\Gamma_{\pm\mu} = 1 \pm \gamma_\mu$, and κ is the hopping parameter. For non-zero μ $\det W$ and S are complex and direct MC is not possible. We still have

$$\det W(\mu) = [\det W(-\mu)]^* \tag{5}$$

4. Approaches and methods of solution

The difficulties connected with the complex action manifest themselves as a number of problems, most notorious:

- The *sign problem*: the results depend directly or indirectly (dependig on the approach) on strong cancellations between contributions of opposite sign both in the values of the the observables and, more noxious, in the partition function (which is the normalization of the measure).
- The *overlap problem*: how relevant is the performed importance sampling for the actual problem?
- The *stability problem*: how stable are the calculated expectation values under variation in the parameters of the simulation?
- The *convergence and uniqueness problem* of the procedure by which the final results for the averages are obtained.

Some approaches:

- Map the problem onto a real *effective action* problem using the symmetries of the path integral. (Example: *Alford et al 2001* for the Potts model.)
  A nice method, but model bound. The effective action is typically non local, the method depends on the possibility of identifying classes of configurations, defining clusters, etc. Once this is achieved there only remain the usual problems of statistical errors, slowing down, etc.

- Use Canonical Ensemble formulation and/or *functional transforms* to redefine the problem in terms of a number of simpler partition functions. (*Karsch, Azcoiti, De Forcrand, ...*).
  The method may acknowledge the sign problem and the convergence problem concerning the decomposition and resummation.

- Perform simulations at values of the parameters for which the action is real and continue by *analytic expansion* to the complex action. (*De Forcrand, Lombardo, Myamura et al, Ejiri et al*)
  The method can be affected by stability questions concerning the extrapolation from data with statistical errors, and by convergence and uniqueness problems leading to systematic errors.

- Perform simulations with a different action and obtain the results by *reweighting*.
  The method suffers of the sign and overlap problems.

- Use a *complex stochastic process* such as the complex Langevin Equation controlled by the actual action.
  The method is subject to the convergence and uniqueness problem.

Notice that all approaches imply at some moment a numerical simulation, redefined such as to be applicable, and/or supplemented by other procedures which may amplify also the usual uncertainties.

5. A reweightying study of the HDM model.

(*De Pietri, Feo, Seiler, IOS*)

The QCD grand canonical partition function (Wilson fermions):

$$Z = \int DU\, \mathrm{e}^{-S}, \quad S = S_{YM} - \log\det W \tag{6}$$

$$\begin{aligned} W &= 1 - \kappa \sum_{i=1}^{3} \left(\Gamma_{+i} U_{x,i} T_i + \Gamma_{-i} U^{\dagger}_{x,i} T_{-i} \right) \\ &\quad -\kappa\gamma \left(\mathrm{e}^{\mu} \Gamma_{+4} U_{x,4} T_4 + \mathrm{e}^{-\mu} \Gamma_{-4} U^{\dagger}_{x,4} T_{-4} \right) \end{aligned} \tag{7}$$

Here T are lattice translations, $\Gamma_{\pm\mu} = 1 \pm \gamma_\mu$, and κ is the hopping parameter $\sim 1/M$. γ is a (bare) anisotropy parameter. For non-zero μ $\det W$ and S are complex. The temperature is introduced as

$$aT = \frac{\gamma}{N_\tau} \tag{8}$$

Hoping parameter expansion for a systematic approximation:

$$\begin{aligned}
\mathrm{Det}\, W &= \exp(\mathsf{Tr} \ln W) \qquad (9)\\
&= \exp\left[-\sum_{l=1}^{\infty} \sum_{\{\mathcal{C}_l\}} \sum_{s=1}^{\infty} \frac{\left(\kappa_\lambda^l g_{\mathcal{C}_l}\right)^s}{s} \mathsf{Tr}_{\mathrm{D,C}} \mathcal{L}_{\mathcal{C}_l}^s \right]\\
&= \prod_{l=1}^{\infty} \prod_{\{\mathcal{C}_l\}} \mathrm{Det}_{\,\mathrm{D,C}} \left(1 \;-\; (\kappa_\lambda)^l g_{\mathcal{C}_l} \mathcal{L}_{\mathcal{C}_l}\right)
\end{aligned}$$

with $\mathcal{C}_l$ a closed, non-self-repeating path, λ the links on $\mathcal{C}_l$ and

$$\mathcal{L}_{\mathcal{C}_l} = \left(\prod_{\lambda \in \mathcal{C}_l} \Gamma_\lambda U_\lambda \right)^s, \; g_{\mathcal{C}_l} = \left(\epsilon\, \mathrm{e}^{\pm N_\tau \mu_f}\right)^r \text{ or } 1 \qquad (10)$$

with non-trivial $g_{\mathcal{C}_l}$ for loops winding r times in the ± 4 direction with periodic(antiperiodic) b.c. $(\epsilon = +1(-1))$ and $\kappa_\lambda = \kappa$ or $\kappa\,\gamma$ for spatial/temporal links.

Large mass, large chemical potential limit (*Bender et al 1992*):

$$\kappa \to 0,\ \mu \to \infty, \quad \kappa\, \mathrm{e}^{\mu} \equiv \zeta :\ \text{fixed} \tag{11}$$

0-th order:

$$\begin{aligned} \mathcal{Z}_F^{[0]}(C,\{U\}) &= \exp\left[-2\sum_{\{\vec{x}\}}\sum_{s=1}^{\infty}\frac{(\epsilon C)^s}{s}\ \mathsf{Tr}(\mathcal{P}_{\vec{x}})^s\right] \\ &= \prod_{\{\vec{x}\}} \mathrm{Det}\ (\mathbf{1}\ -\ \epsilon\, C\mathcal{P}_{\vec{x}})^2, \quad C = (2\,\zeta)^{N_\tau}\ , \end{aligned} \tag{12}$$

($\mathcal{P}_{\vec{x}}$: Polyakov loops).

NB: If one wants to ensure the symmetry $\det W(\mu) = [\det W(-\mu)]^*$ one can take also non-dominant terms in the determinant, thus

$$\mathcal{Z}_F^{[0]} = \prod_{\vec{x}} \det\left(1 - \epsilon h \mathrm{e}^{\frac{\mu}{T}}\mathcal{P}_{\vec{x}}\right)^2 \det\left(1 - \epsilon h \mathrm{e}^{-\frac{\mu}{T}}\mathcal{P}_{\vec{x}}^{-1}\right)^2, \quad h = (2\kappa)^{N_\tau}$$

2-nd order:

$$\mathcal{Z}_F^{[2]}(\kappa,\mu,\{U\}) = \exp\left\{ -2 \sum_{\{\vec{x}\}} \sum_{s=1}^{\infty} \frac{(\epsilon\, C)^s}{s} \times \right.$$
$$\left. \times\ \mathrm{Tr}\left[\left(\mathcal{P}_{\vec{x}} \right)^s + \kappa^2 \sum_{r,q,i,t,t'} (\epsilon\, C)^{s(r-1)} \left(\mathcal{P}^{r,q}_{\vec{x},i,t,t'} \right)^s \right] \right\}$$
$$= \mathcal{Z}_F^{[0]}(C,\{U\}) \prod_{\vec{x},r,q,i,t,t'} \mathrm{Det}\left(\mathbf{1} - (\epsilon\, C)^r\, \kappa^2\, \mathcal{P}^{r,q}_{\vec{x},i,t,t'} \right)^2 . \quad (13)$$

Use temporal gauge for easy bookkeeping. The model is defined in terms of the "Boltzmann factor" (*Aarts et al 2002*)

$$B = \mathrm{e}^{-S_{YM}}\, \mathcal{Z}_F^{[2]} \ .$$

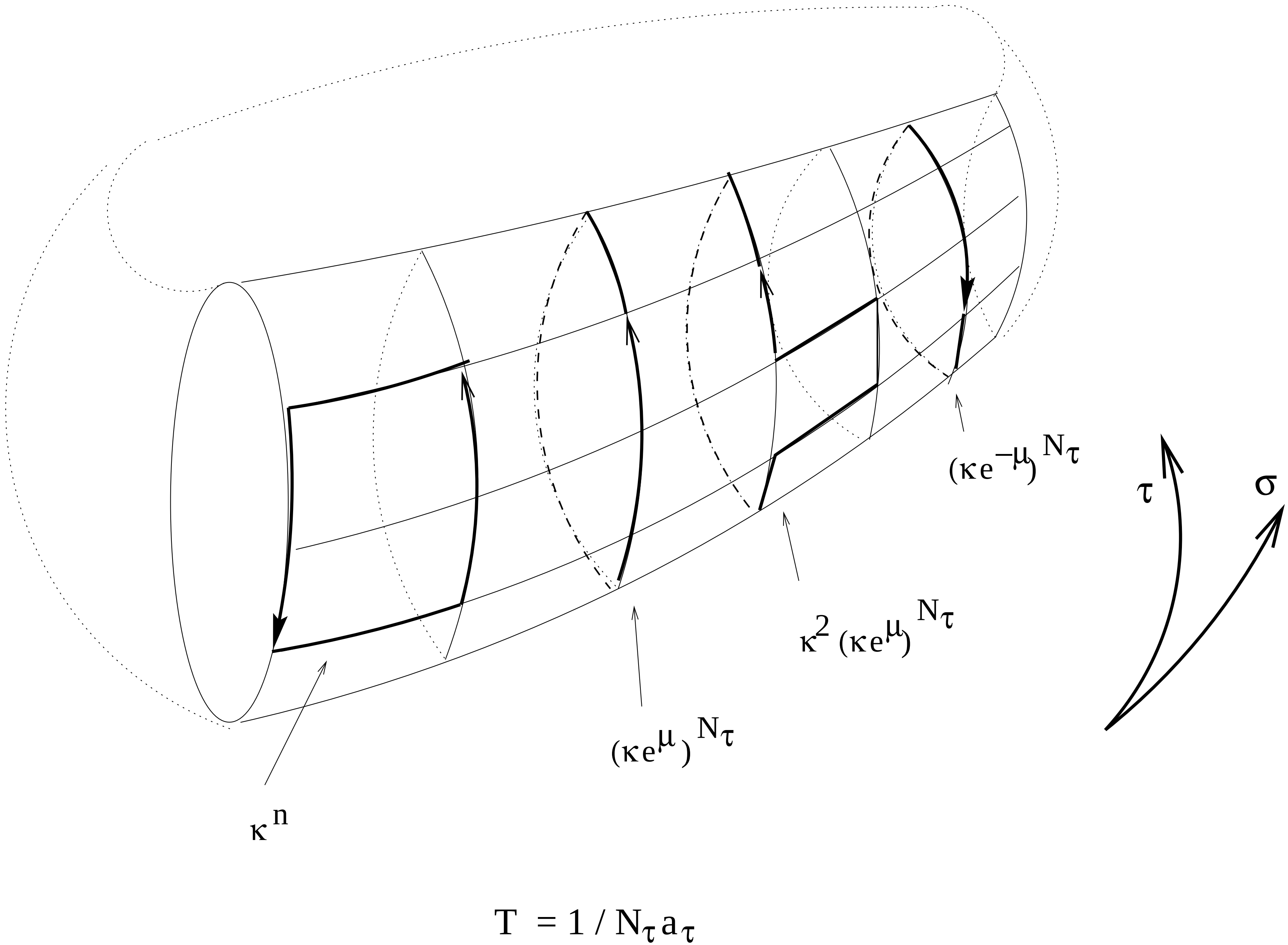


Figure 2: Periodic lattice, loops, temporal gauge.

Analytic results:

- strong coupling, hopping parameter, leading behaviour of the Polyakov loop and its adjoint:

$$\begin{aligned} \langle P \rangle &\sim C^2 \left(1 + \frac{4}{9} \beta \kappa^2 (N_\tau - 1) \right), \\ \langle P^* \rangle &\sim \frac{2}{3} C \left(1 + \frac{1}{3} \beta \kappa^2 (N_\tau - 1) \right) \end{aligned}$$

- anisotropic mean field (using γ to define the temperature).

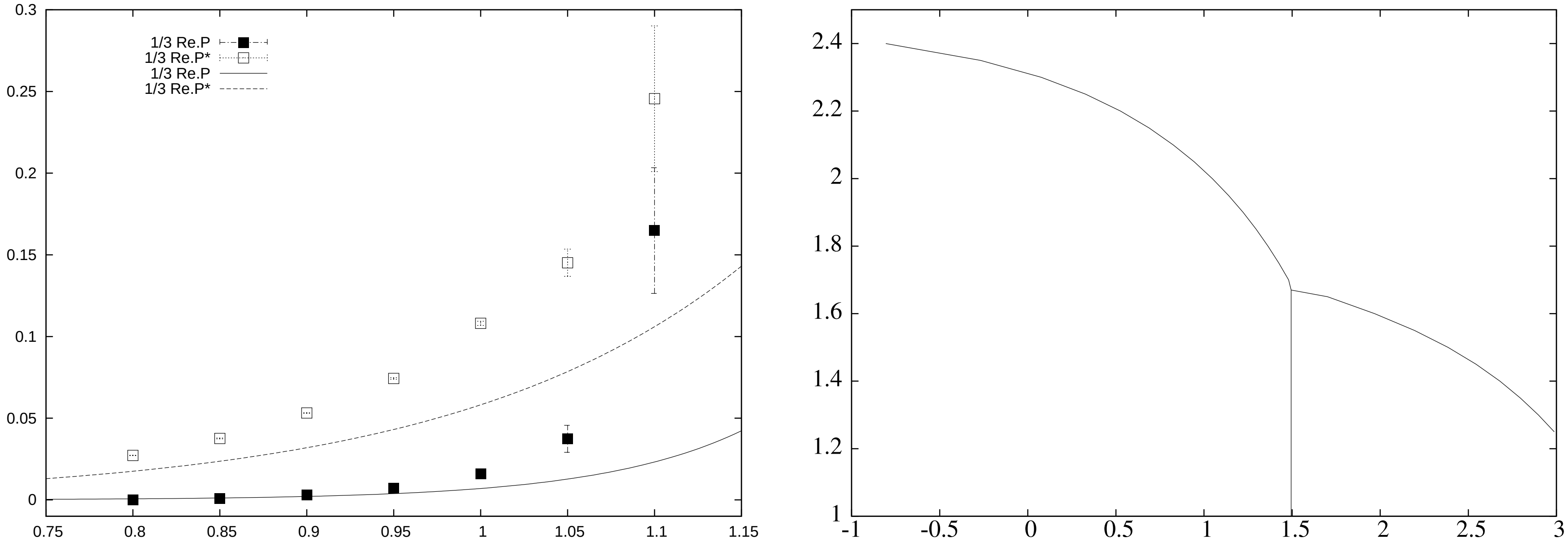


Figure 3: Left: Comparison with strong coupling, $\beta = 5.5$, 6^4 lattice. Symbols are the reweighting results. Right: Mean field phase diagram (abscissa μ, ordinate $\gamma = N_\tau \, a \, T$).

Reweighting analysis (3 flavours, 6^4 lattices, $\sim 10^7$ sweeps per point):

split a positive definite Boltzman factor B_0 from B and define the weights w and the expectation values

$$B = B_0 w\,, \quad \langle O \rangle = \frac{\langle w O \rangle_0}{\langle w \rangle_0} \tag{14}$$

with

$$\begin{aligned} B_0 &\equiv \prod_{Plaq} \mathrm{e}^{\frac{\beta}{3} Re \mathsf{Tr} Plaq} \times \prod_{\vec{x}} \mathrm{e}^{2C Re \mathsf{Tr}\left[\mathcal{P}_{\vec{x}} + \kappa^2 \sum_{i,t,t'} \mathcal{P}^{0,1}_{\vec{x},i,t,t'}\right]}\,, \\ w &\equiv \prod_{\vec{x}} \mathrm{e}^{-2\,C\,Re \mathsf{Tr}\left[\mathcal{P}_{\vec{x}} + \kappa^2 \sum_{i,t,t'} P^{0,1}_{\vec{x},i,t,t'}\right]} \times \mathcal{Z}_F^{[2]}\,. \end{aligned} \tag{15}$$

We measure Polyakov loops, baryonic density, their susceptibilities, etc.

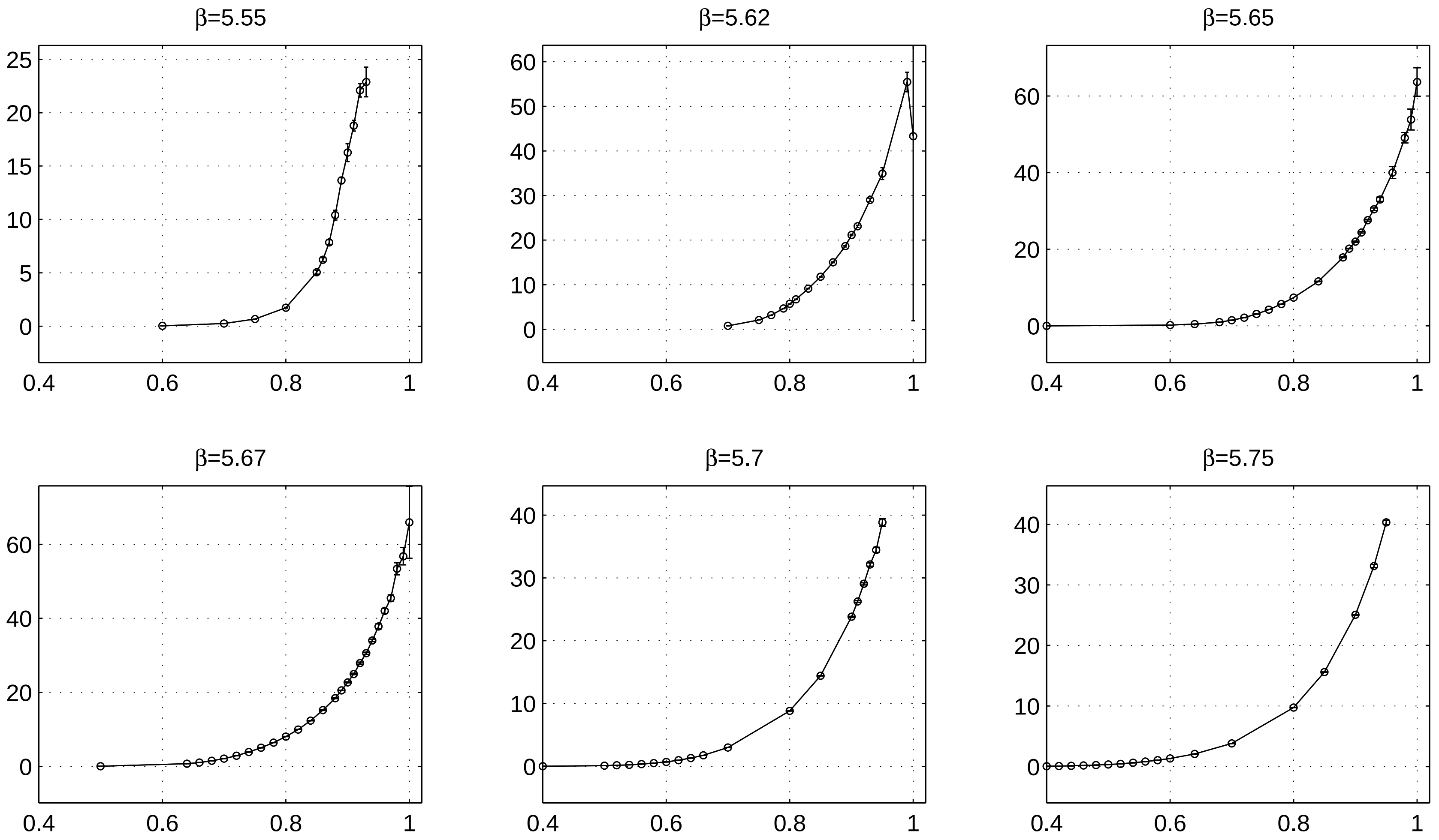


Figure 4: Baryonic density vs. μ at fixed β.

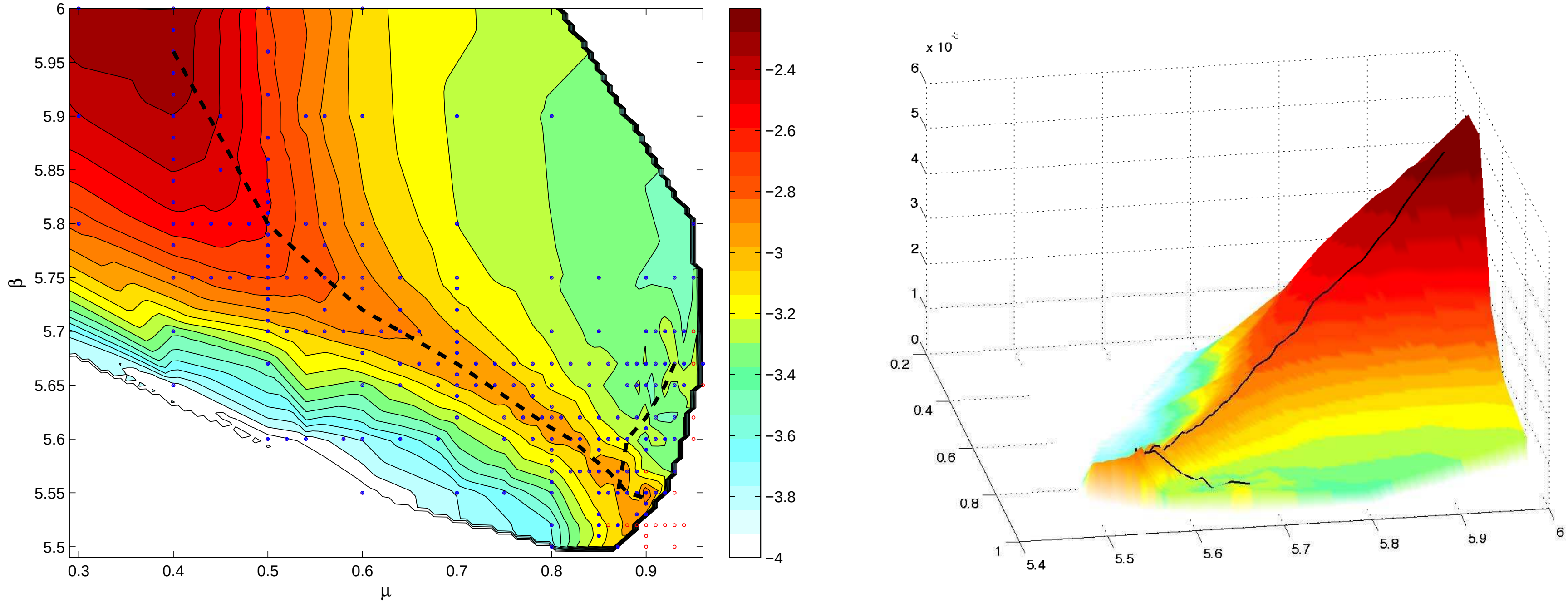


Figure 5: Landscape of the Polyakov loop susceptibility over the $\beta - \mu$ plane (increasing β means larger temperature). The color scale is based on $\log_{10}(\chi_P)$.

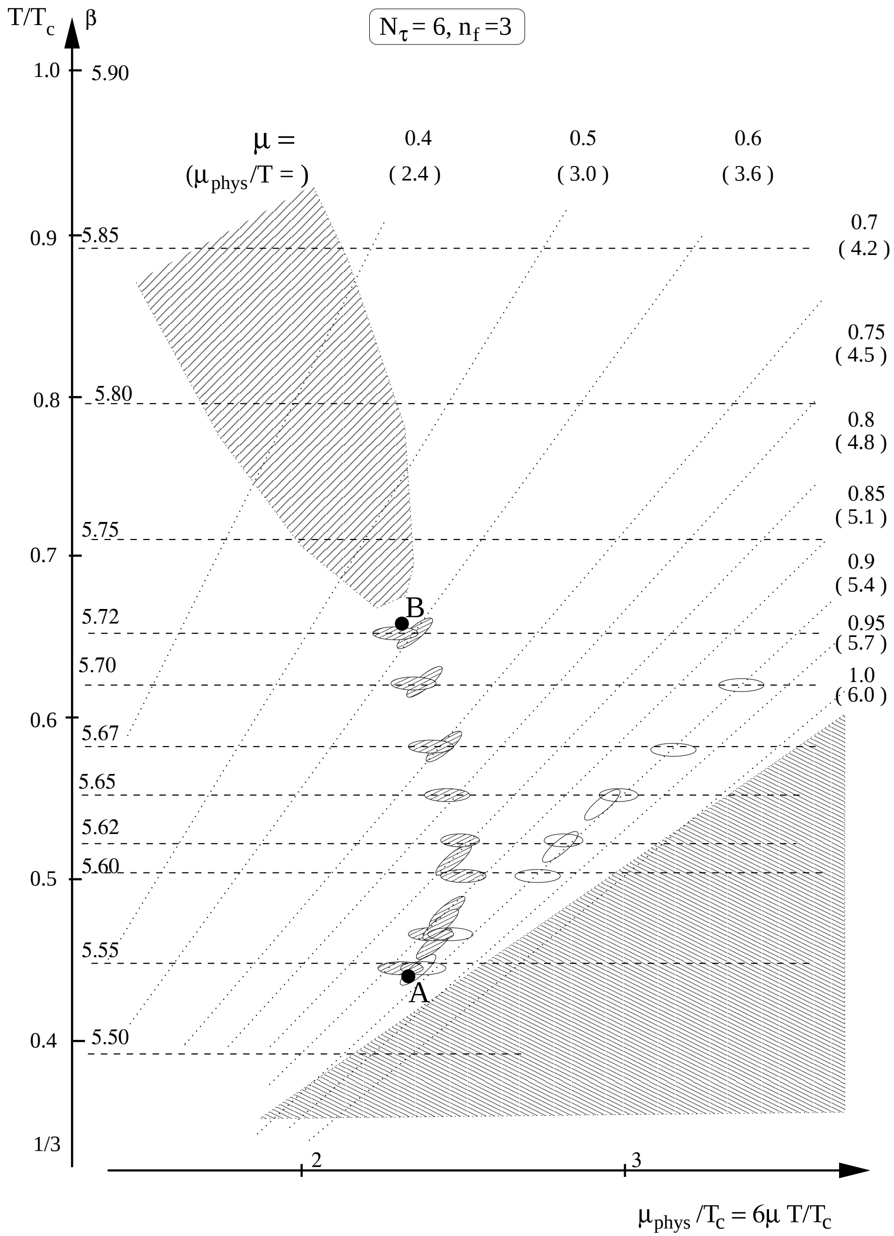


Figure 6: Tentative phase diagram.

Further insight can be gained from the distribution of the values of the Polyakov loop in different phases:

- true histogram (dependent on the choice of B_0):

$$H_\Delta(x,y) = \left\langle \Theta_{\Delta,x}\left(\frac{Re(w\,P_{\vec{x}})}{\langle w\rangle_0}\right)\,\Theta_{\Delta,y}\left(\frac{Im(w\,P_{\vec{x}})}{\langle w\rangle_0}\right)\right\rangle_0 \tag{16}$$

with $\Theta_{\Delta,s}(t) = 1$ if $|t-s| \leq \Delta/2$, 0 otherwise.

- complex "distribution" (independent on the choice of B_0):

$$T_\Delta(x,y) = \langle \Theta_{\Delta,x}(ReP_{\vec{x}})\,\Theta_{\Delta,y}(ImP_{\vec{x}})\rangle\ , \tag{17}$$

and we have:

$$\langle P\rangle \approx \sum_{x,y}(x+iy)T_\Delta(x,y)\,, \tag{18}$$

Also, the weight distribution changes.

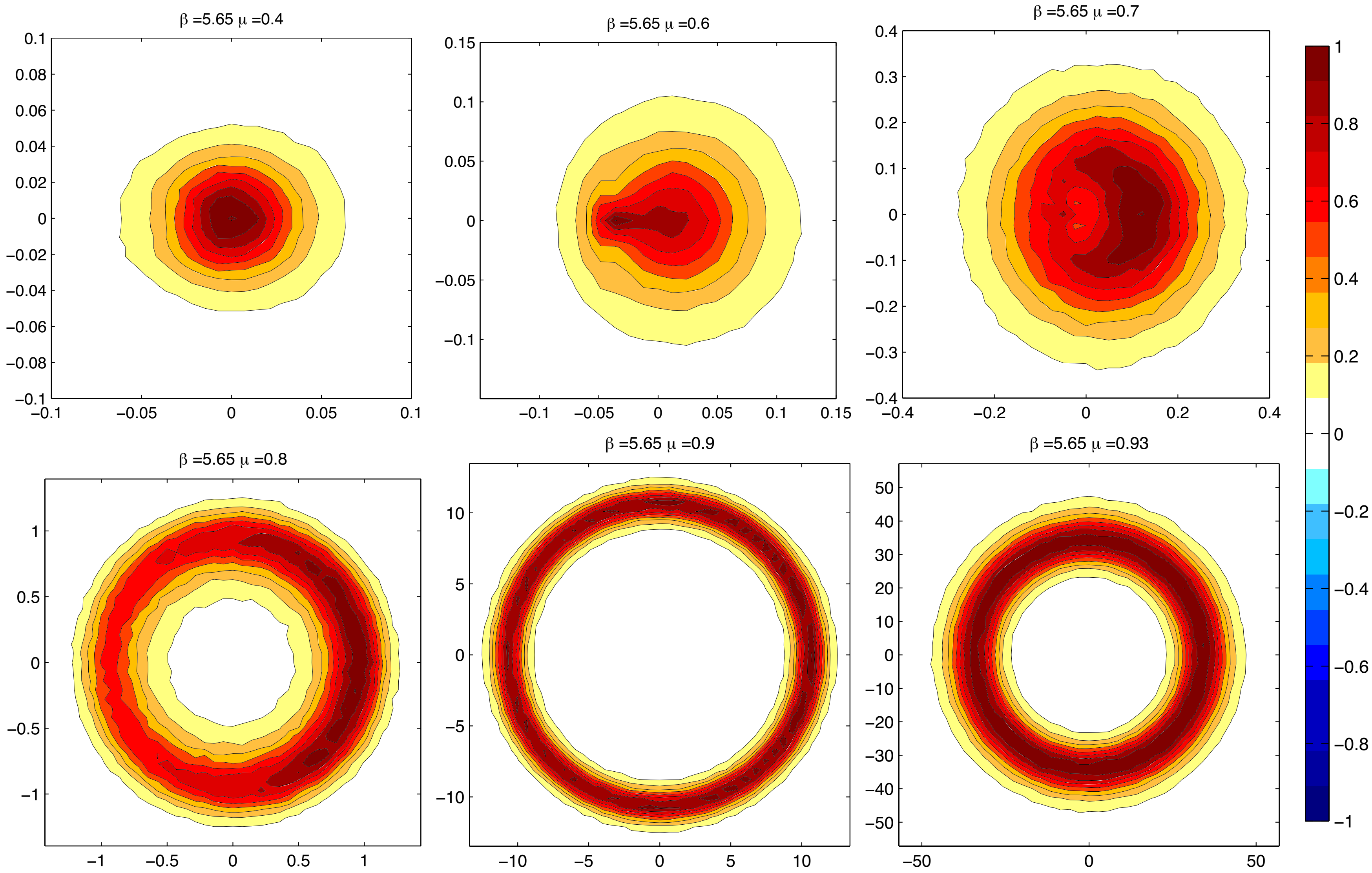


Figure 7: Polyakov loop 'histogram' $H_{\Delta}(x, y)$ of eq. (16) vs. μ at $\beta = 5.65$.

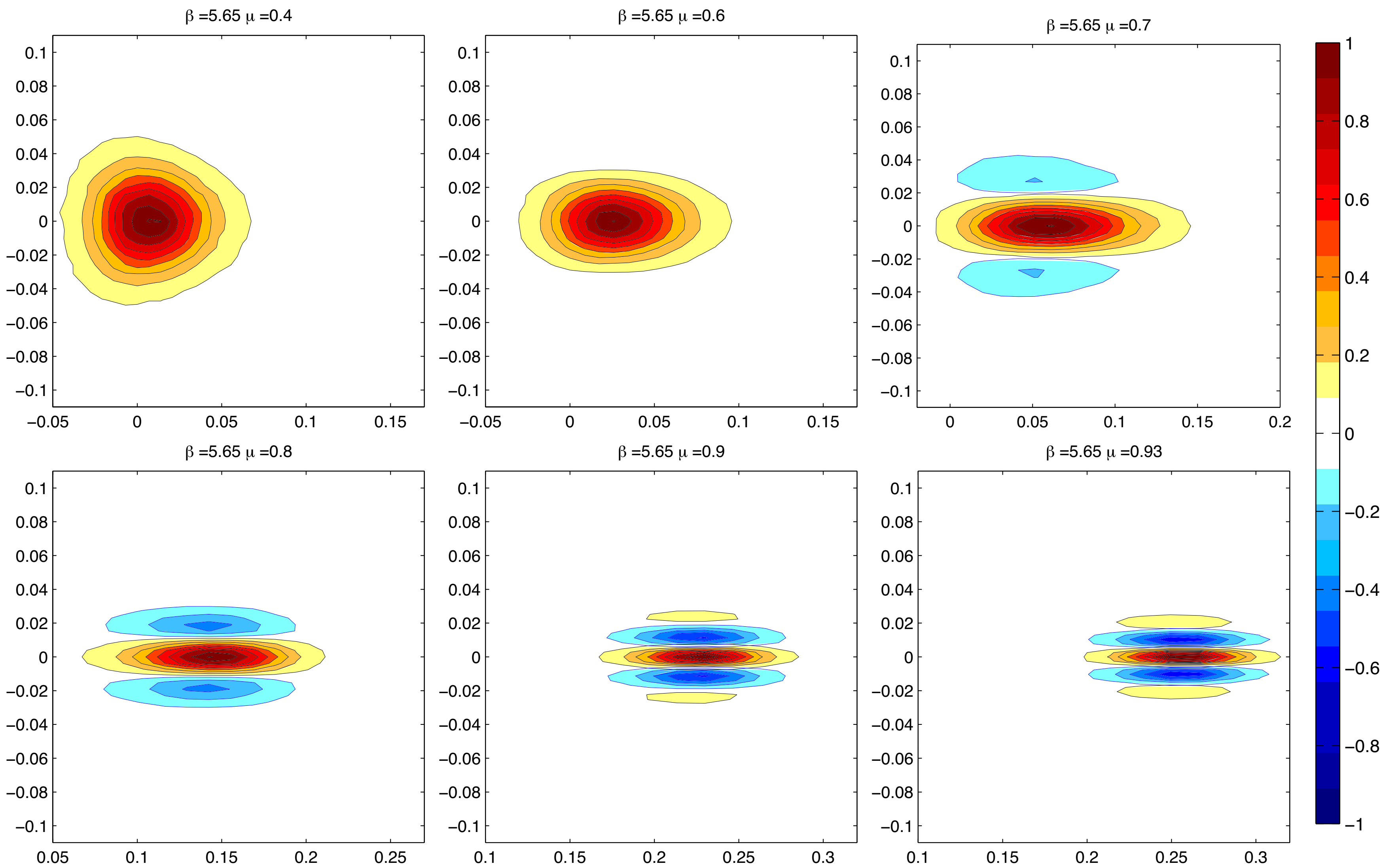


Figure 8: Real part of the Polyakov loop ‘distribution’ $T_\Delta(x,y)$ of eq. (17) vs. μ at $\beta = 5.65$ fixed.

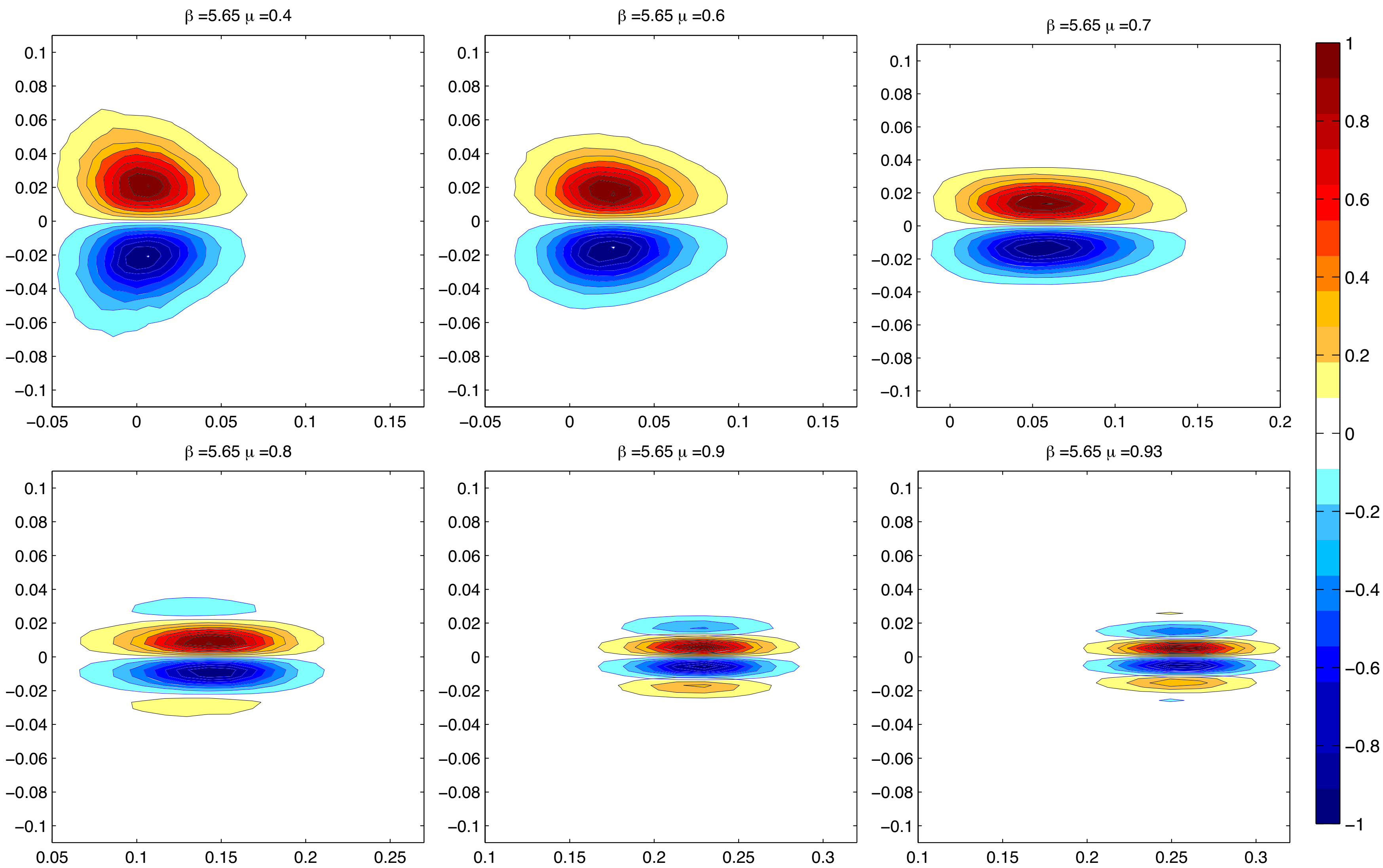


Figure 9: Imaginary part of the Polyakov loop 'distribution' $T_\Delta(x, y)$ of eq. (17) vs. μ at $\beta = 5.65$ fixed.

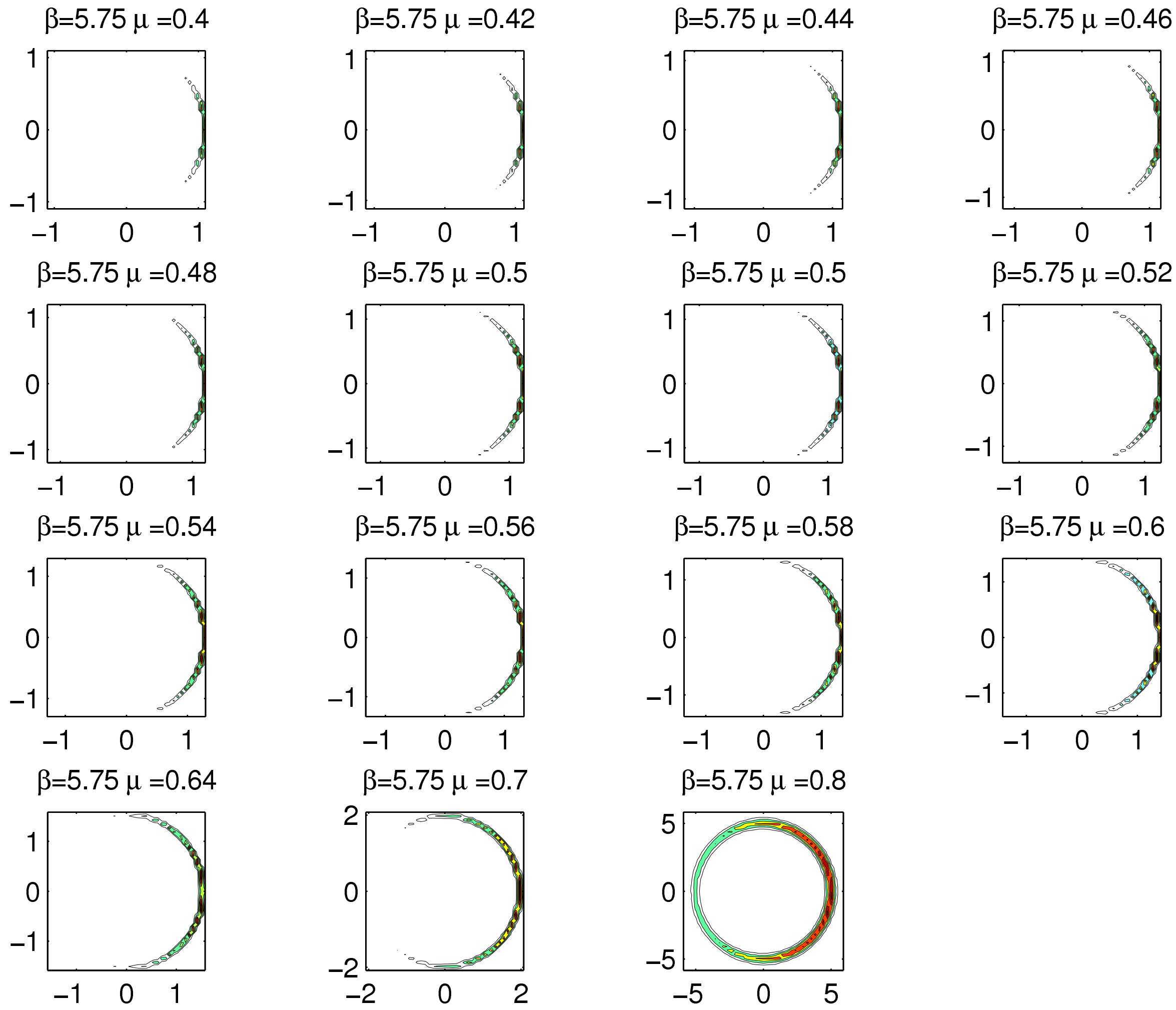


Figure 10: Weight factor w 'distribution' vs. μ at $\beta = 5.75$ fixed.

6. Stochastic processes for complex action.

a) General aspects.

b) Simple examples and what can we learn from them.

c) Application to Minkowski QFT problems.

d) Application to chemical potential problems.

Stochastic processes in Quantum Field Theory

Procedure: Realize a sampling of field configurations by defining a supplementary (noisy) dynamics in a 5-th "time".

Basic example: Parisi Wu stochastic quantization in Euclidean QFT

- proofs of equivalence with path integral formulation, proofs of convergence, etc
  rely on the definition of a probability distribution over the space of field configurations via an associated Fokker-Planck equation
- can define a "perturbation theory" without gauge fixing
- for numerical studies: comparable to MC

  (see also Damgaard and Hueffel, Namiki, ...)

Essential feature: uses a drift force to define the process (the *5-th time dynamics*)

$\longrightarrow$ **versatility**

- can be directly related to expectation values
- can be directly defined from the set up of the problem without needing an action or a probability interpretation for the path integral

This may be of interest in cases where other approaches (e.g., MC) do not work.

In the following: point of view of numerical simulations.

Usual realizations: Langevin Equation and Random Walk.

Here in discretized form, Ito calculus, ϑ: 5-th "time", $\delta\vartheta$: "time" step; for a field $\varphi(x)$ (random variable), $K[\varphi]$: drift force,

Langevin equation:

$$\delta\varphi(x;\vartheta) \equiv \varphi(x;\vartheta+\delta\vartheta) - \varphi(x;\vartheta) = K[\varphi(x;\vartheta)]\,\delta\vartheta + \eta(x;\vartheta)$$

$$\langle\eta(x;\vartheta)\rangle = 0, \quad \langle\eta(x;\vartheta)\eta(x';\vartheta')\rangle = 2\,\delta\vartheta\,\delta_{x,x'}\,\delta_{\vartheta,\vartheta'}$$

Random Walk:

$$\delta\varphi(x;\vartheta) = \pm\omega, \quad \text{with pbb}: \ \tfrac{1}{2}(1 \pm \tfrac{1}{2}\omega\,K[\varphi(x;\vartheta)])\,, \ \omega = \sqrt{\delta\vartheta}$$

NB: since η, $\omega \propto \sqrt{\delta\vartheta}$ we need also second derivatives:

$$\delta f[\varphi(\vartheta] = \partial_\varphi f[\varphi(\vartheta)]\,\delta\varphi(x;\vartheta) + \tfrac{1}{2}\partial_\varphi^2 f[\varphi(\vartheta)]\,[\delta\varphi(x;\vartheta)]^2$$

Relation to path integral and MC

If the drift is the gradient of a real action, bounded from below **then** there is a probability density $P(\varphi, \vartheta)$ satisfying an associated Fokker-Planck Equation in the limit $\delta\vartheta \longrightarrow 0$:

$$\partial_\vartheta P(\varphi, \vartheta) = \partial_\varphi \left(\partial_\varphi - K\right) P(\varphi, \vartheta), \quad K = -\partial_\varphi S$$

and we have:

$$P(\varphi, \vartheta) = c_0 \mathrm{e}^{-S(x)} + \sum_{E_n > 0} c_n \phi_n \mathrm{e}^{-E_n t} \rightarrow P_{as}(\varphi) = c_0 \mathrm{e}^{-S[\varphi]}, \; (\vartheta \rightarrow \infty)$$

with E_n the *eigenvalues* of the Fokker–Planck Hamiltonian:

$$H_{FP} = -\partial_\varphi^2 + \frac{1}{4}(\partial_\varphi S)^2 - \frac{1}{2}(\partial_\varphi^2 S)$$

- expectation values $\langle f(\varphi)\rangle$ can be calculated as averages over the noise, equivalently as ϑ averages:

$$\overline{f(\varphi)} = \frac{1}{\Theta}\int_0^\Theta d\vartheta\, f(\varphi(\vartheta)) = \langle f(\varphi)\rangle + \mathcal{O}(1/\sqrt{\Theta})\,,$$

- the convergence is controlled by the properties of the FP Hamiltonian,

- in practice $\delta\vartheta \neq 0$: $\rho_{as}(\varphi)$ has $\mathcal{O}(\delta\vartheta)$ corrections (controllable).

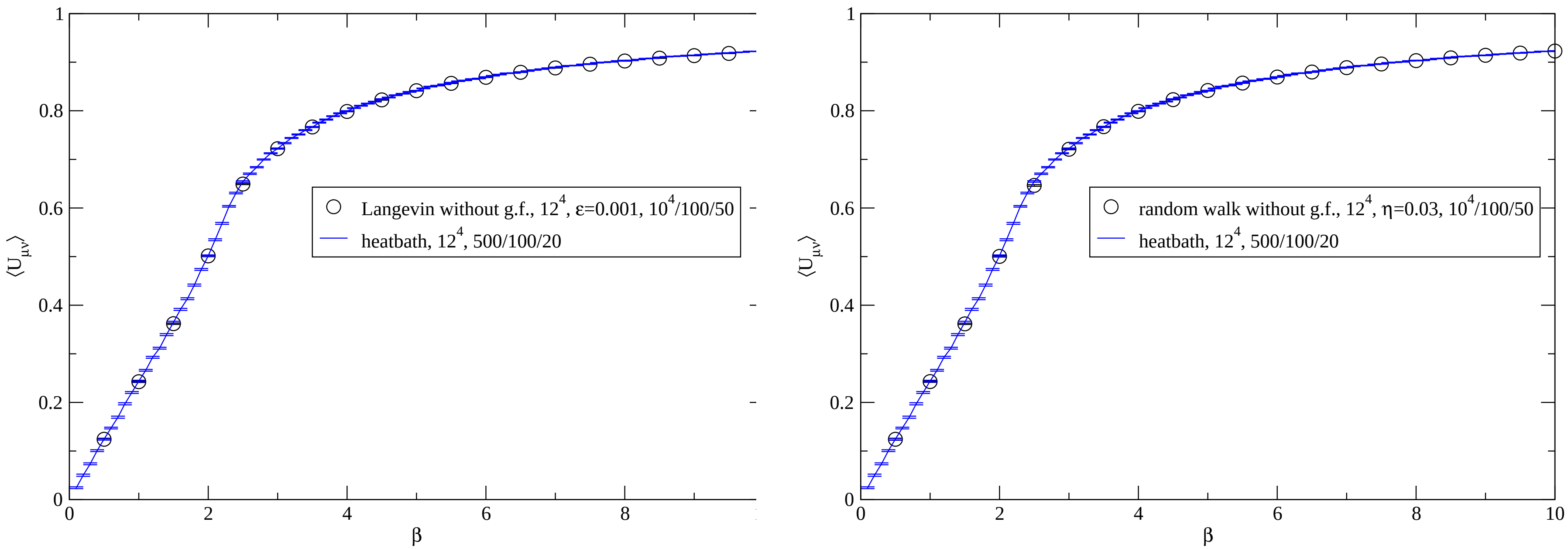


Figure 11: Plaquette averages by LE and RW compared with MC

Developments based on the versatility of the method:

- redefining the drift force,
- changing the action,
- redefining the noise (e.g., nonlinear processes: "active brownian motion"),
- ...

Very much used in modeling (statistical physics, complex systems, etc.)
In QM and QFT: open systems, continuous localization, special simulation problems.

Interesting if Euclidean formulation is not possible or ambiguous.

In the following: Complex action.

Complex action

- reformulation of the stochastic quantization for real time evolution
- accounting for complex terms in the action (non-zero density)
- in both cases: (more or less formal) proofs of convergence and equivalence to the path integral formulation under certain conditions (*many papers since the 1980's: Paris, ... Hüffel and Rumpf, Okamoto, etc*)

The complex process

Since the drift is complex the process automatically provides an imaginary part for the variable. Hence we must define the process in the complex plane, i.e., we must complexify each degree of freedom.

LE with complex drift $K(z)$ for a complex variable $z(\vartheta) = x(\vartheta) + i\, y(\vartheta)$ amounts to two related real LE with independent noise terms

$$\begin{aligned}
\delta z(\vartheta) &= K(z,\vartheta)\,\delta\vartheta + \eta(\vartheta)\,, \quad \eta = N_R\,\eta_R + \mathrm{i}\,N_I\,\eta_I && (19)\\
\text{i.e.}: \quad & \delta x(\vartheta) = Re\,K(z,\vartheta)\,\delta\vartheta + N_R\,\eta_R(\vartheta) && (20)\\
& \delta y(\vartheta) = Im\,K(z,\vartheta)\,\delta\vartheta + N_I\,\eta_I(\vartheta) && (21)\\
\langle \eta_R \rangle &= \langle \eta_I \rangle = 0\,, \quad \langle \eta_R^2 \rangle = \langle \eta_I^2 \rangle = 2\,\delta\vartheta\,, \; \langle \eta_R \eta_I \rangle = 0\,, \quad N_R - N_I = 1
\end{aligned}$$

The noise normalizations N_R, N_I are defined such as to reproduce the correct quantum fluctuations, ortherwise they are arbitrary.

For the two real variables we can also define a corresponding pair of real RW processes

$$\delta x(\vartheta) = \pm\omega_x\,, \quad P_{x,\pm} = \tfrac{1}{2}\,(1 \pm \frac{\omega_x}{2N_R} Re\, K(z,\vartheta)) \tag{22}$$

$$\delta y(\vartheta) = \pm\omega_y\,, \quad P_{y,\pm} = \tfrac{1}{2}\,(1 \pm \frac{\omega_y}{2N_I} Im\, K(z,\vartheta)) \tag{23}$$

$$\omega_x = \sqrt{2N_R\delta\vartheta}\,,\ \omega_y = \sqrt{2N_I\delta\vartheta} \tag{24}$$

where $P_\pm$ are the transition probabilities and we have defined the steps such as to have the same $\delta\vartheta$ in all processes, to ensure the correct correlation.

We can define a FPE for a complex "distribution" $P(z)$, z : complex

$$\partial_\vartheta P(z,\vartheta) = \partial_z \left(\partial_z - K\right) P(z,\vartheta), \quad K = -\partial_z S$$

Formally this has as asymptotic solution e^{-S}.

Alternatively we can define a genuine probability distribution $\rho(x,y,t)$ for the real variables x, y from the system (20,21) or (22,23). Using, e.g., the master equation with the transition probabilities (22,23) we can write for it a real FPE:

$$\partial_\vartheta \rho(x,y,\vartheta) = \left[\partial_x \left(N_R\, \partial_x - ReK(z)\right) + \partial_y \left(N_I\, \partial_y - ImK(z)\right)\right] \rho(x,y,\vartheta),$$

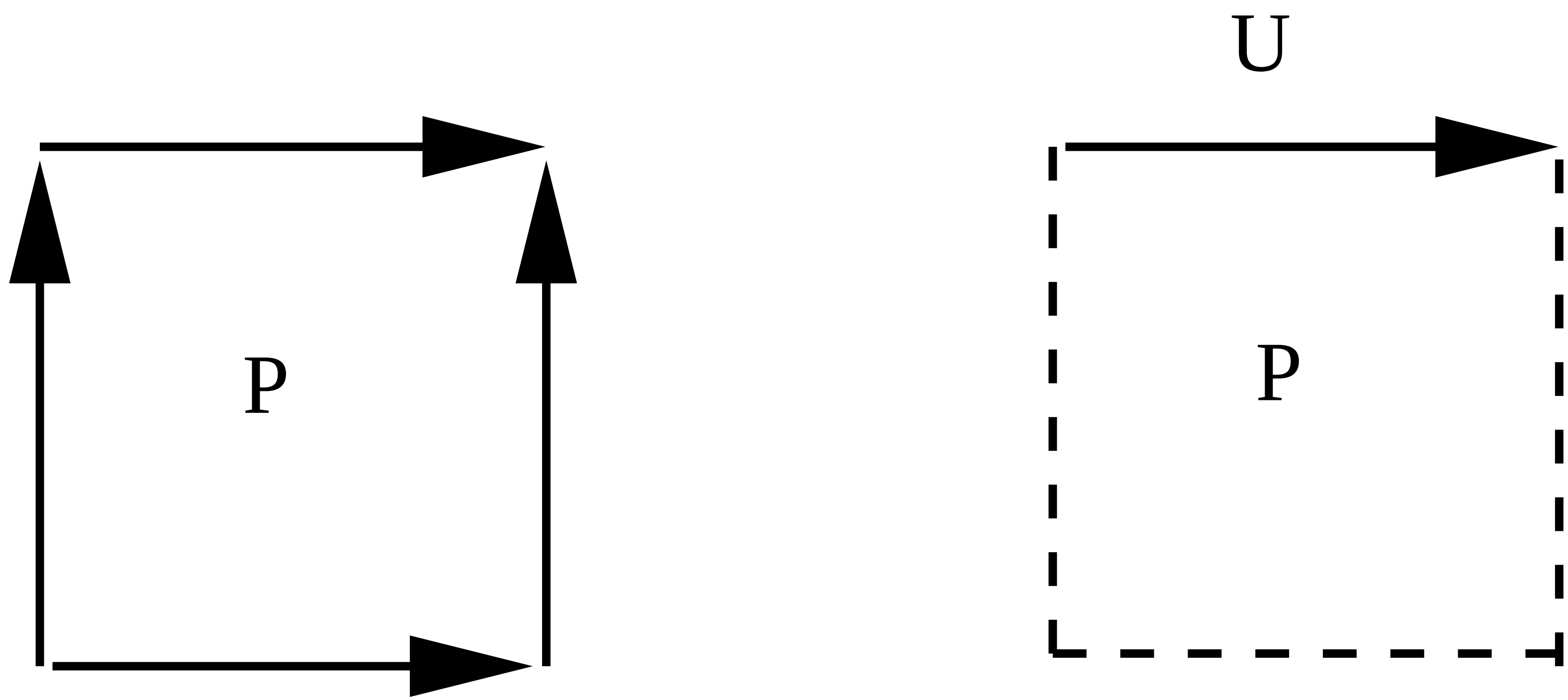
For analytic $f(z)$ we have

$$\int dx\, dy\, \rho(x,y,\vartheta)\, f(x+iy) = \int dx\, P(x,\vartheta)\, f(x) \qquad (25)$$

The relation to convergence is more subtle.

Simple models

One plaquette with complete gauge fixing $\longrightarrow$ one link integrals.
($Berges,\ Sexty,\ Seiler,\ Aarts,\ IOS$)



One link U(1) model

$$
\begin{aligned}
Z &= \int dU\, \mathrm{e}^{-S_{YM}+i\,p\,x} \det W = \int_{-\pi}^{\pi} \frac{dx}{2\pi}\, \mathrm{e}^{-S(x)}, \qquad (26)\\
S_{YM} &= -\frac{\beta}{2}\left(U+U^{-1}\right) = -\beta \cos x\,, \quad U = \mathrm{e}^{i\,x}\\
\det W &= 1+\frac{\kappa}{2}\left[\mathrm{e}^{\mu} U + \mathrm{e}^{-\mu} U^{-1}\right] = 1+\kappa\cos(x-i\mu).
\end{aligned}
$$

Observables (exact expressions in terms of Bessel functions)

$\langle U\rangle = \langle \mathrm{e}^{ix}\rangle\,,\ \langle U^{-1}\rangle = \langle \mathrm{e}^{-ix}\rangle$ (Polyakov loop and its inverse)

$\langle \cos x\rangle = \frac{\partial}{\partial\beta}\ln Z$ (plaquette), $\langle n\rangle = \frac{\partial}{\partial\mu}\ln Z = \left\langle \frac{i\kappa\sin(x-i\mu)}{1+\kappa\cos(x-i\mu)}\right\rangle$

Notice that in complexifying the variable $x \longrightarrow z = x + i\,y$ we must use U^{-1} instead of U^* everywhere. p is a "reweighting" parameter.

For $\kappa = 0$, $\beta : imaginary$ we moke Minkowski quantization

For $\beta : real$ and κ, $\mu \neq 0$ we moke the chemical potential (HDM)

“Minkowski” case

The Langevin process can be made to converge at $p \neq 0$ and using reweighting to calculate the averages also at $p = 0$. General criteria obtained from the analysis of the classical flow ($Berges,\ Sexty$).

The complex FPE for the complex “distribution” $P_p(x)$

$$\dot{P}_p(x, \vartheta) \;=\; \left(\partial_x^2 - i\,\partial_x\,(p + \beta\,\sin x)\right)\,P_p(x, \vartheta) \tag{27}$$

can be easily solved for the Fourier modes. It shows good convergence (depending on p) and the results agree with the averages from the LE process ($Aarts,\ Seiler,\ IOS$). The analysis of the spectrum of the FPE shows that in this region the $eigenvalues$ are positive. Here we used real noise ($N_R = 1, N_I = 0$).

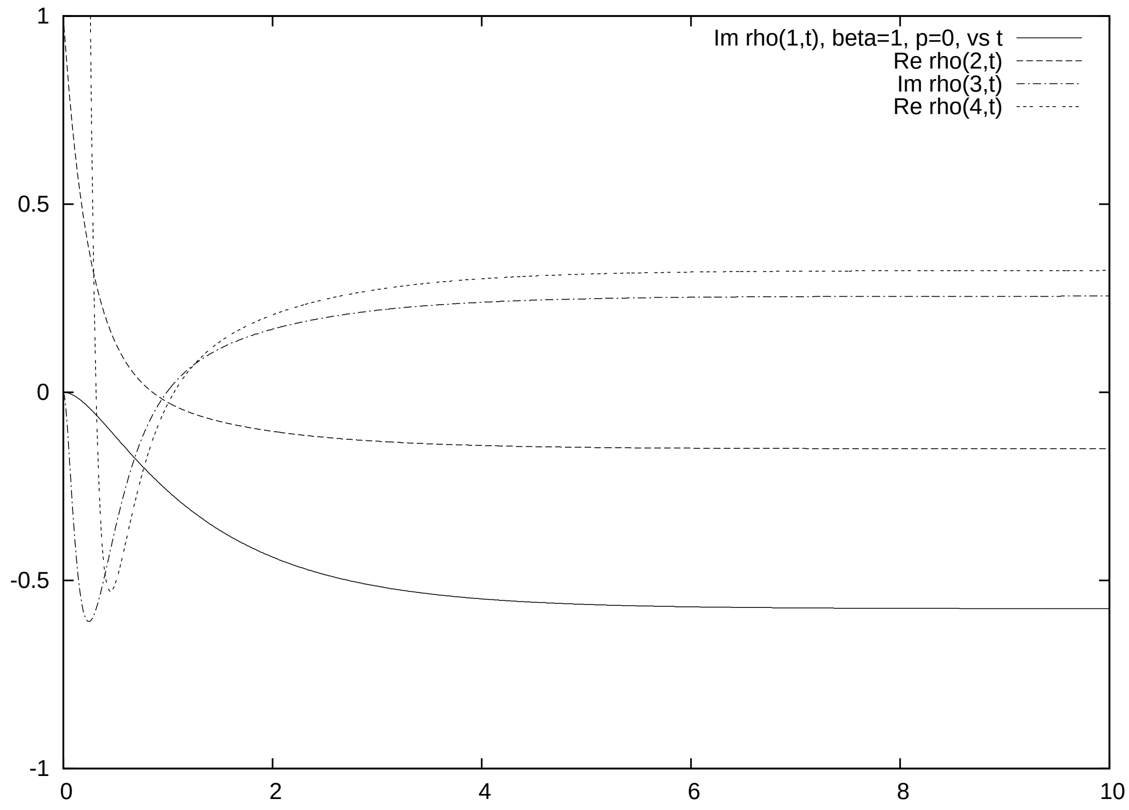


Figure 12: $P_p(k, \vartheta)$ (Re, Im) for $p = 0$ and $\beta = 1$, $k = 1, 2, 3, 4$ from the complex FPE.

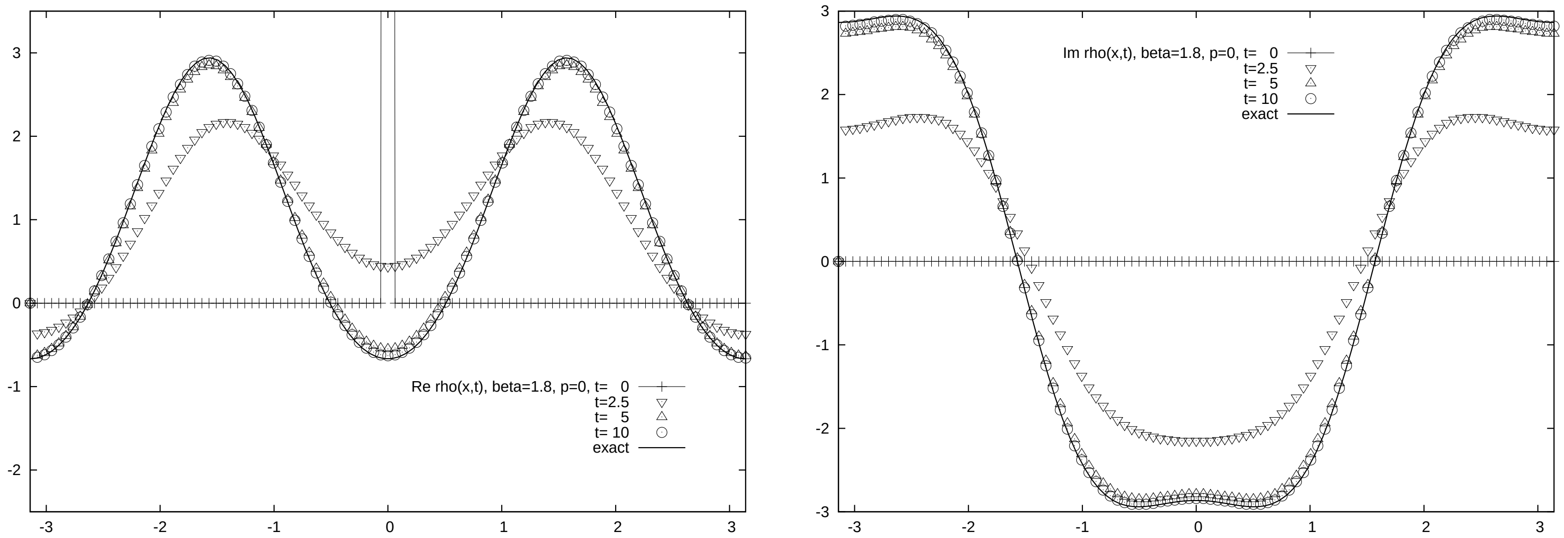


Figure 13: $P_0(x, \vartheta)$ (Re, Im), $p = 0$, $\beta = 1.8$ with $N = 100$ x-discretization, $\delta\vartheta = 10^{-6}$, for $t = 0, 2.5, 5, 10$ compared with exact measure e^{-S} (discretized).

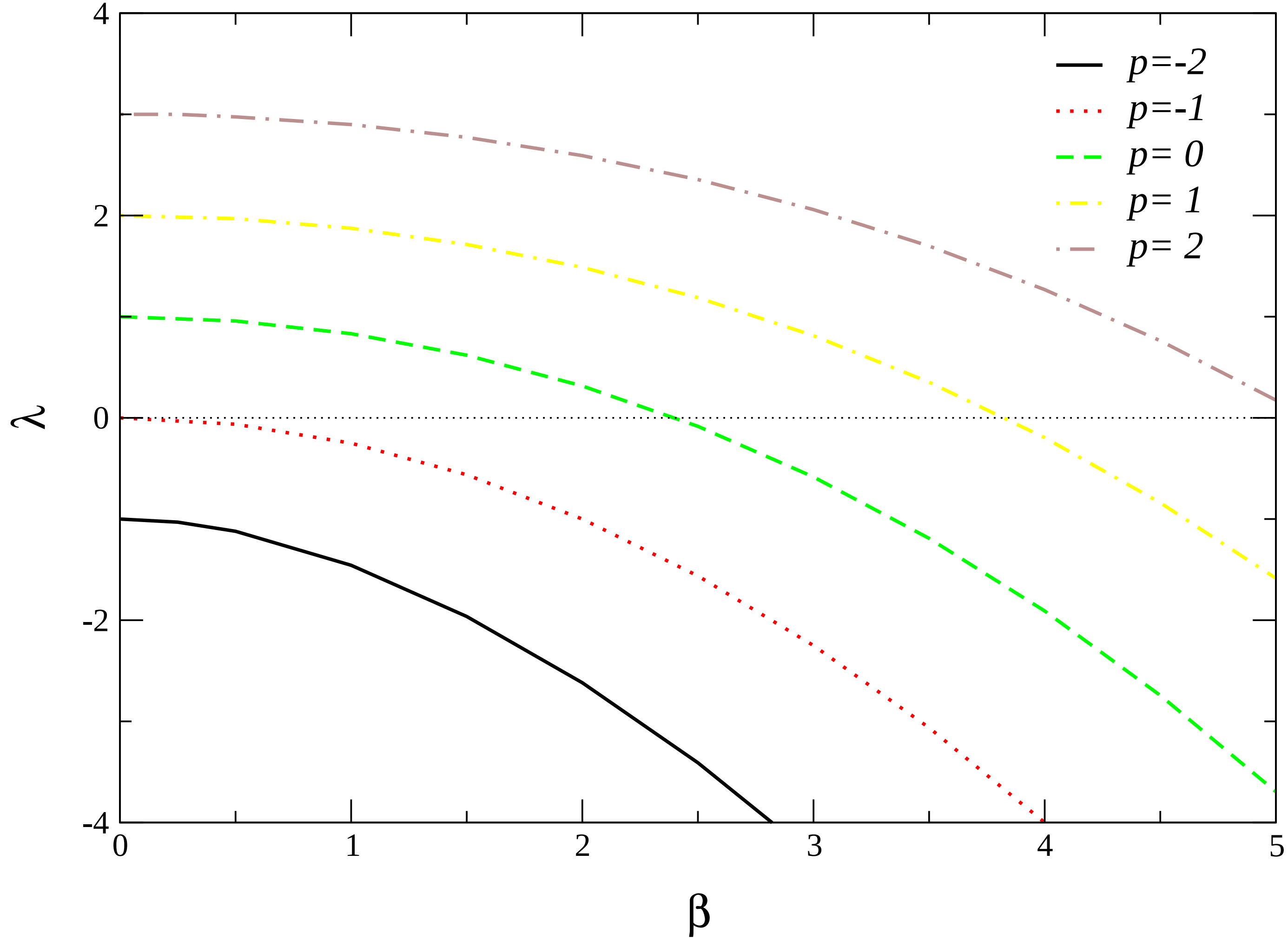


Figure 14: Lowest *eigenvalues* of the FP Hamiltonian for imaginary β. This also illustrates the role of p.

"'Chemical potential" (symmetric HDM)

Here we take $p = 0$. The Langevin process converges everywhere ($Aarts,\ Seiler,\ IOS$).

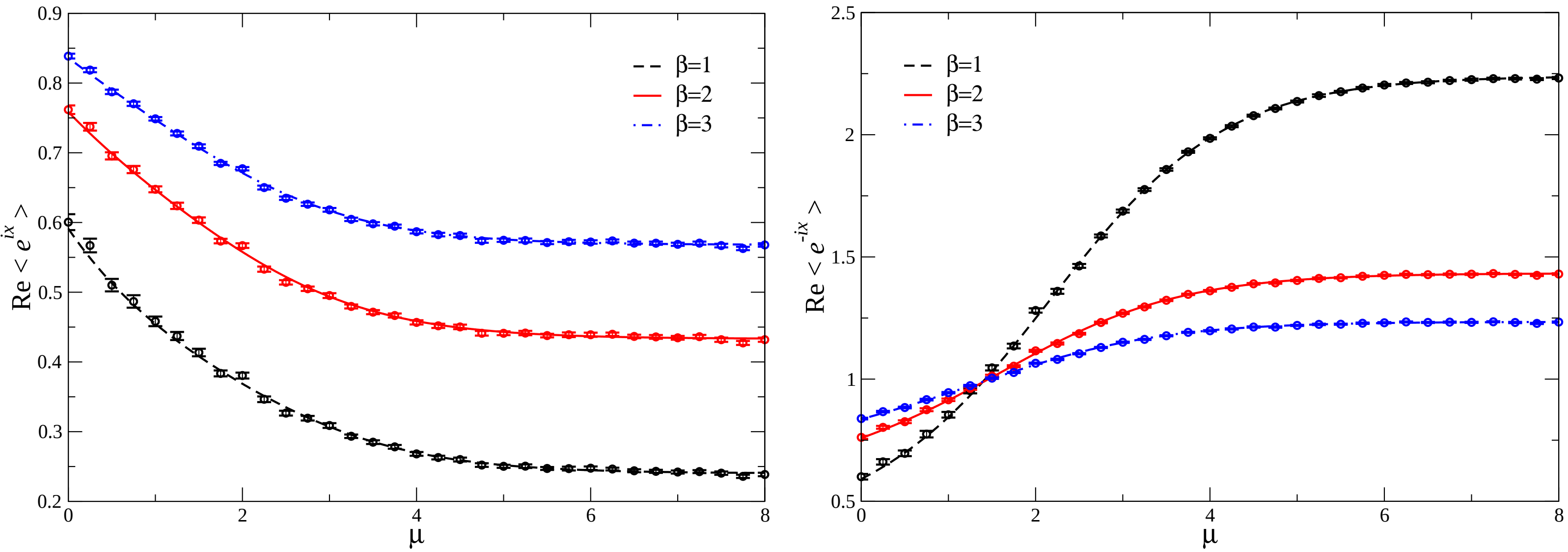
The complex FPE for the modes of the complex "distribution" $P(x)$

$$\dot{P}(n, \vartheta) = -n^2 P(n, \vartheta) - nc_+ P(n+1, \theta) + nc_- P(n-1, \vartheta) \tag{28}$$

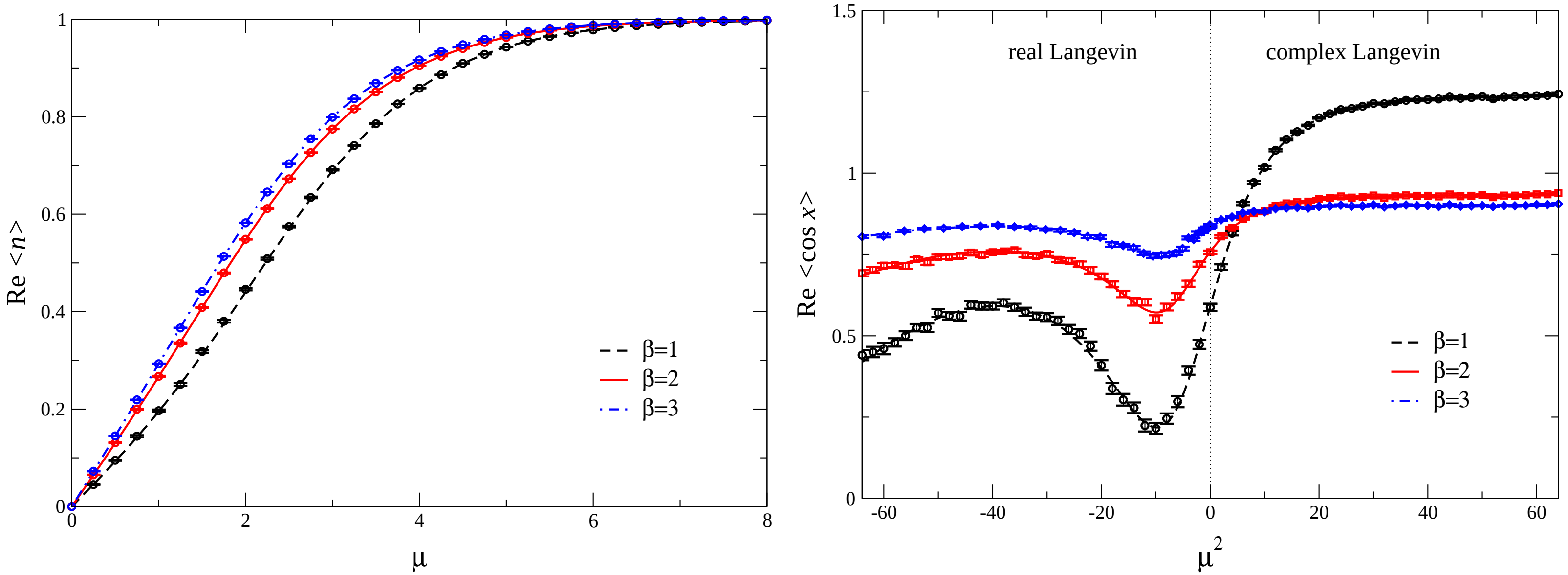
with $c_\pm = \frac{1}{2}\left(\beta + \kappa \mathrm{e}^{\pm\mu}\right)$ can be easily solved and shows good convergence toward $\exp(-S)$. The results agree with the averages from the LE process. This behaviour is supported by the analysis of the spectrum of the FPE Hamiltonian and the fixed point structure of the classical flow.

Here we used again real noise ($N_R = 1, N_I = 0$).

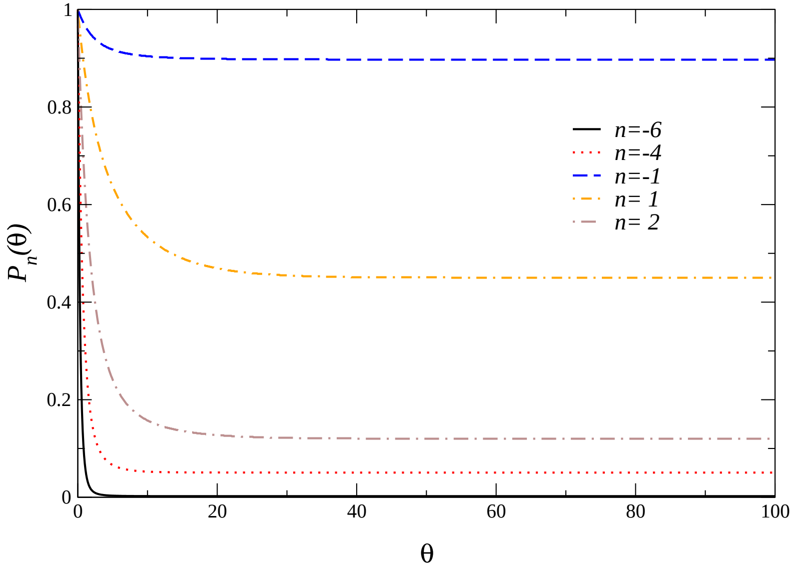
U(1), Polyakov loop and its inverse vs μ

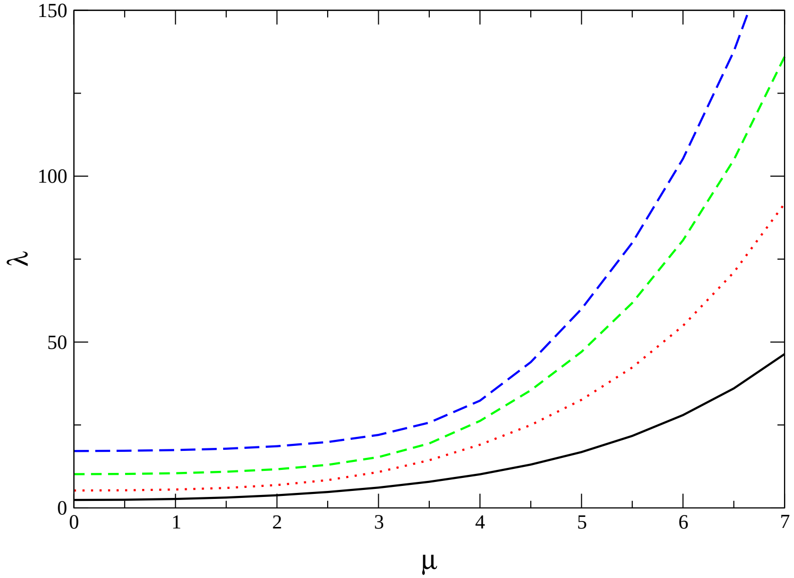


U(1), density vs μ, plaquette vs μ^2

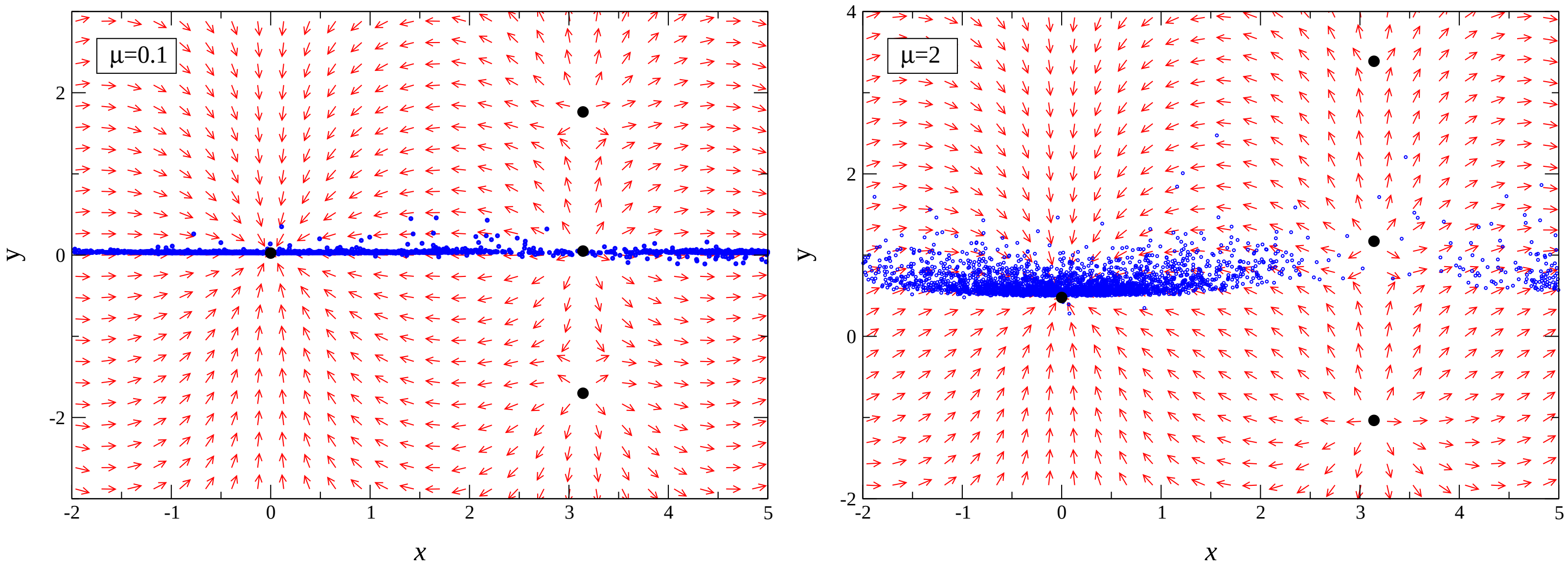


U(1), Lowest eigenvalues of the FP Hamiltonian ($\beta = 1$, $\kappa = 0.5$)





U(1), Classical flow of the drift (below: $\beta = 1$, $\kappa = 0.5$ and $\mu = 0.1$ and 2).

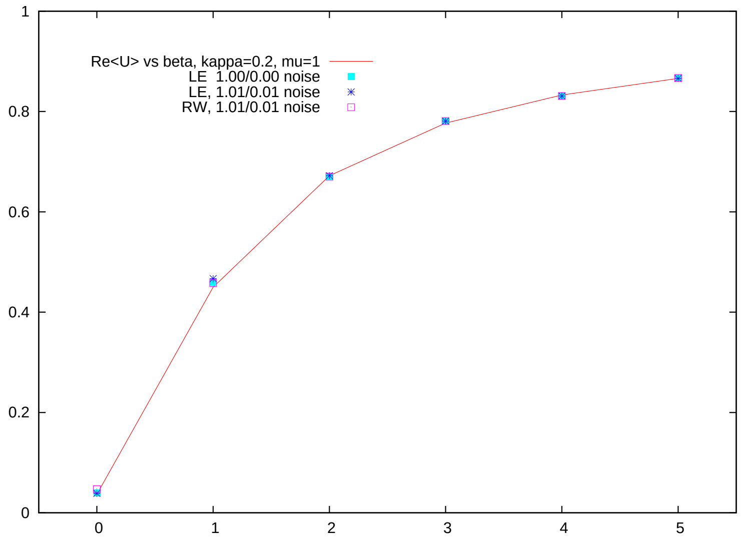


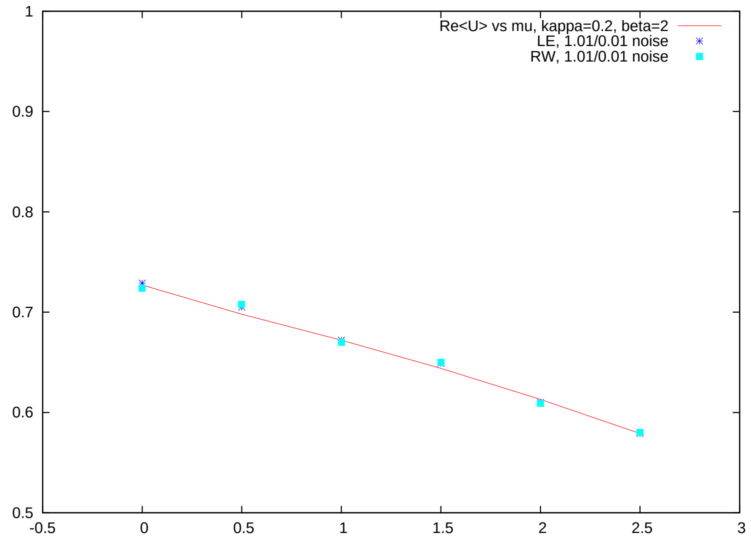
For this problem we also have written down the RW *process*. In this case *we cannot take purely real noise*. In the following we use $N = 1.01 + \mathrm{i}\,0.01$.

The agreement with the LE process is very good.

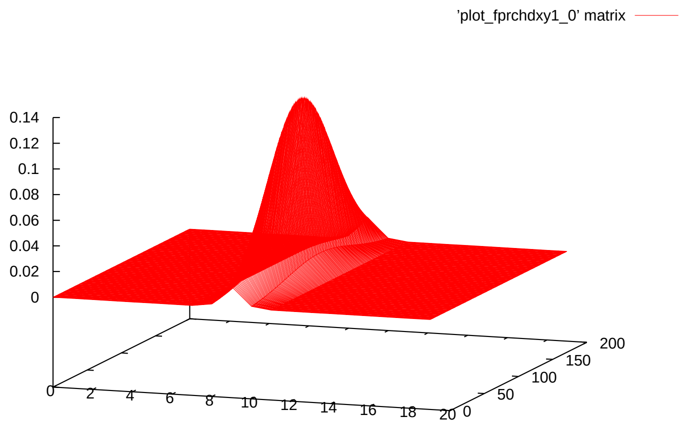
We can also solve the real FPE (for the genuine probability distribution $\rho(x, y, \vartheta)$). Here we took $N = 1.1 + \mathrm{i}\,0.1$.

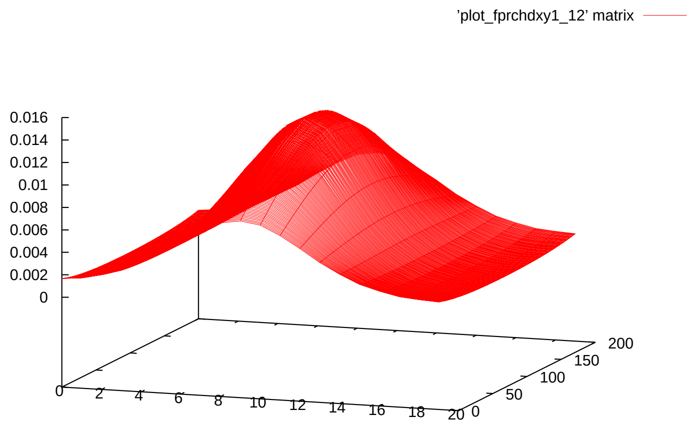
U(1), Plaquette (vs $\beta, \mu = 1$ and vs $\mu,\ \beta = 2,\ \kappa = 0.2$) RW compared with LE and exact.





U(1), Real probability distribution (initial and asymptotic).





One link SU(3) model

Here $U \in SU(3)$ ($\in SL(3,C)$ after complexification),

$$S_{YM} = -\frac{\beta}{6}\left(\mathrm{Tr}U + \mathrm{Tr}U^{-1}\right), \tag{29}$$

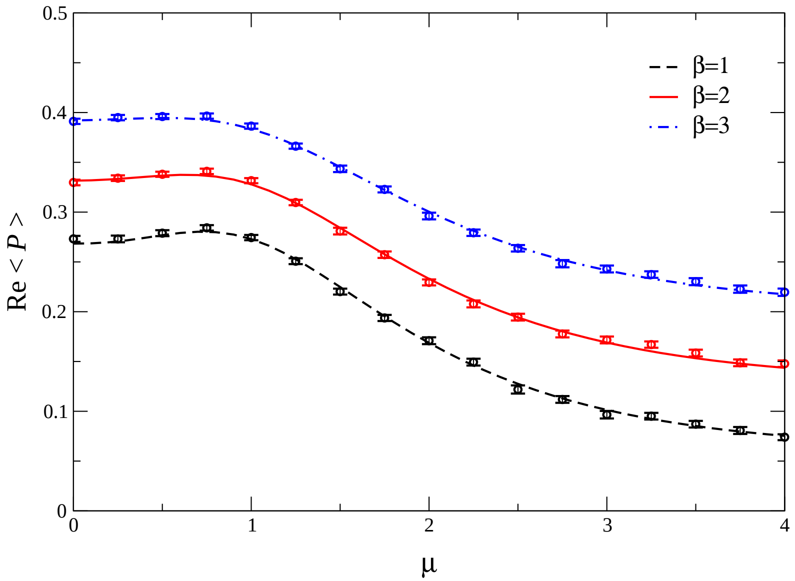
$$\det W = \det\left(1+\kappa e^{\mu}U\right)\det\left(1+\kappa e^{-\mu}U^{-1}\right) \tag{30}$$

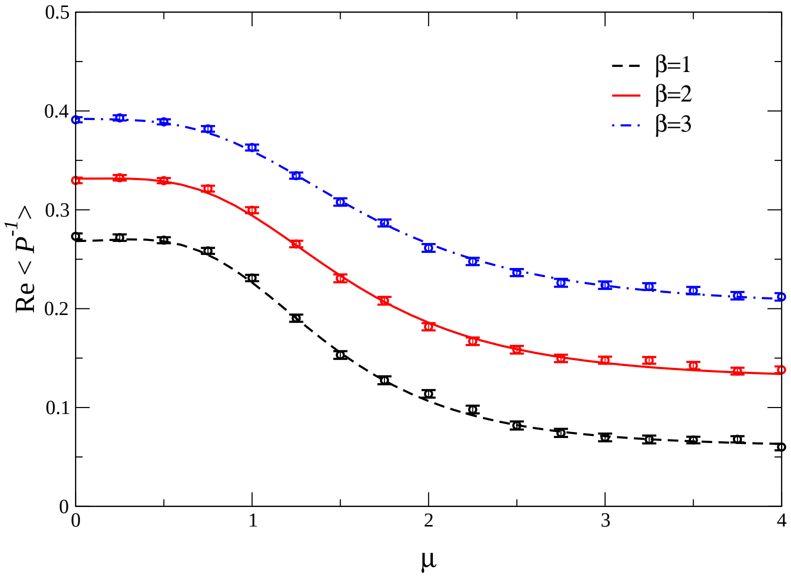
All observables are easily computed exactly using the reduced Haar measure. The LE proceeds in $sl(3,C)$ for the 8 (now complex) components of the potential:

$$U \rightarrow U' = e^{i\lambda_a\left(\epsilon K_a + \sqrt{\epsilon}\eta_a\right)}U \tag{31}$$

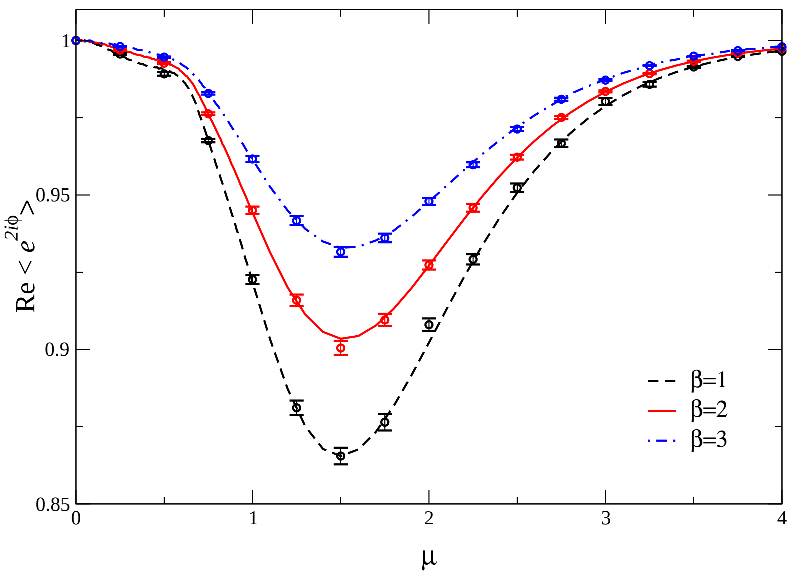
$$K_a = -D_a S, \quad D_a f(U) = \frac{\partial}{\partial\alpha} f\left(e^{i\alpha\lambda_a}U\right)\Big|_{\alpha=0}.$$

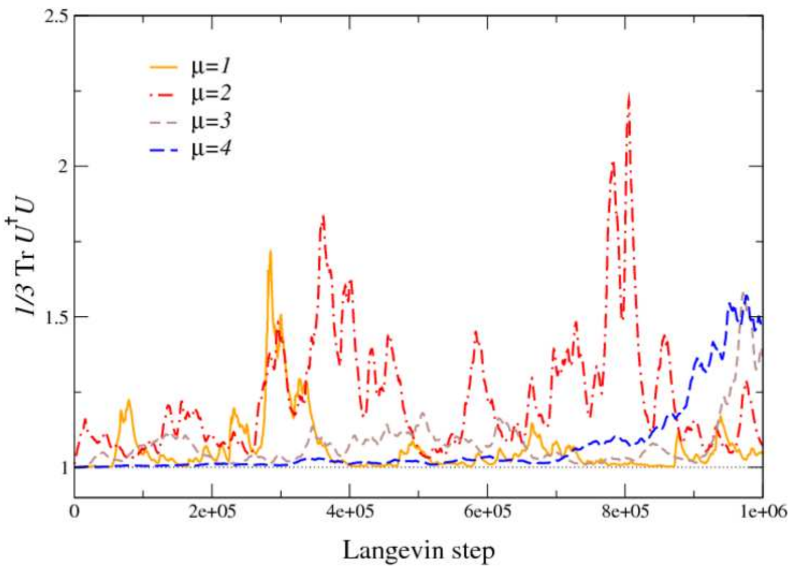
SU(3) Polyakov loop and its inverse vs μ





Phase $\langle \det W(\mu)/\det W(-\mu)\rangle$ for $\kappa = 0.5$ and various β *vs* μ, and $\mathrm{Tr}U^\dagger U/3$ *vs* Langevin step for $\mu = 1, 2, 3, 4$ at $\beta = 1, \kappa = 0.5$





What can we learn from the simple models?

- both LE and RW can be applied

- there are good criteria for convergence (classical flow, FPE spectrum, etc)

- there seems to be no tuning problem, rather we must avoid some special conditions (such as $N_I = 0$ for RW and real FPE) or "repair" them (e.g., use $p > 0$ to force positive $eigenvalues$ in the complex FPE).

- similar results obtain for SU(3) (with complexification SL(3,C)), therefore they are not model dependent (for SU(2) see $Berges\ and\ Sexty$)

- what it is needed is to find equivalent criteria which can be applied to lattice problems

Minkowski QFT

Some recent lattice applications ($Berges,\ Borsanyi,\ Sexty,\ IOS$):

- for scalar field and SU(2) gauge theory,

- finite temperature and non-equilibrium evolution,

- convergence and uniqueness tests,

- relation with the Schwinger-Dyson equations

Scalar field

$\hat{\varphi} = a\varphi$, $\hat{m} = am$, $\hat{\mathbf{x}} = \mathbf{x}/a$, $\hat{t} = t/a_t$, $\gamma = a/a_t$, $\hat{\vartheta} = \vartheta/a^2$, $\epsilon = \delta\vartheta/a^2$,

$\hat{\eta} = \sqrt{a^3 a_t \delta\vartheta}\,\eta = \sqrt{\epsilon/\gamma}\,a^3\eta$, $\quad \langle \hat{\eta}(\hat{x},\hat{\vartheta})\,\hat{\eta}(\hat{x}',\hat{\vartheta}')\rangle_\eta = 2\,\delta_{\hat{x},\hat{x}'}\delta_{\hat{\vartheta},\hat{\vartheta}'}$

$$\begin{aligned}
\hat{\varphi}(\hat{x};\hat{\vartheta}+\epsilon) &= \hat{\varphi}(\hat{x};\hat{\vartheta}) + \sqrt{\epsilon\gamma}\,\hat{\eta}(\hat{x};\hat{\vartheta}) \\
&\quad -i\,\epsilon\left(\Box_\gamma\hat{\varphi}(\hat{x};\hat{\vartheta}) + \hat{m}^2\hat{\varphi}(\hat{x};\hat{\vartheta}) + \lambda\hat{\varphi}(\hat{x};\hat{\vartheta})^3\right). \\
\Box_\gamma\hat{\varphi}(\hat{x};\hat{\vartheta}) &= \gamma^2\left(\hat{\varphi}(\hat{x}+\hat{e}_0;\hat{\vartheta}) + \hat{\varphi}(\hat{x}-\hat{e}_0;\hat{\vartheta}) - c_t\hat{\varphi}(\hat{x};\hat{\vartheta})\right) \\
&\quad -\sum_i\left(\hat{\varphi}(\hat{x}+\hat{e}_i;\hat{\vartheta}) + \hat{\varphi}(\hat{x}-\hat{e}_i;\hat{\vartheta}) - 2\hat{\varphi}(\hat{x};\hat{\vartheta})\right)
\end{aligned}$$

The fields are the complex extensions of the original ones and all observables analytic extensions of the original observables.

One gives *initial conditions* in physical time t, e.g. $\varphi(x,0;\vartheta) = \varphi_0(x)$, $\varphi(x,1;\vartheta) = \varphi(x,0;\vartheta)$ and a *starting configuration* $\varphi(x,t;0)$.

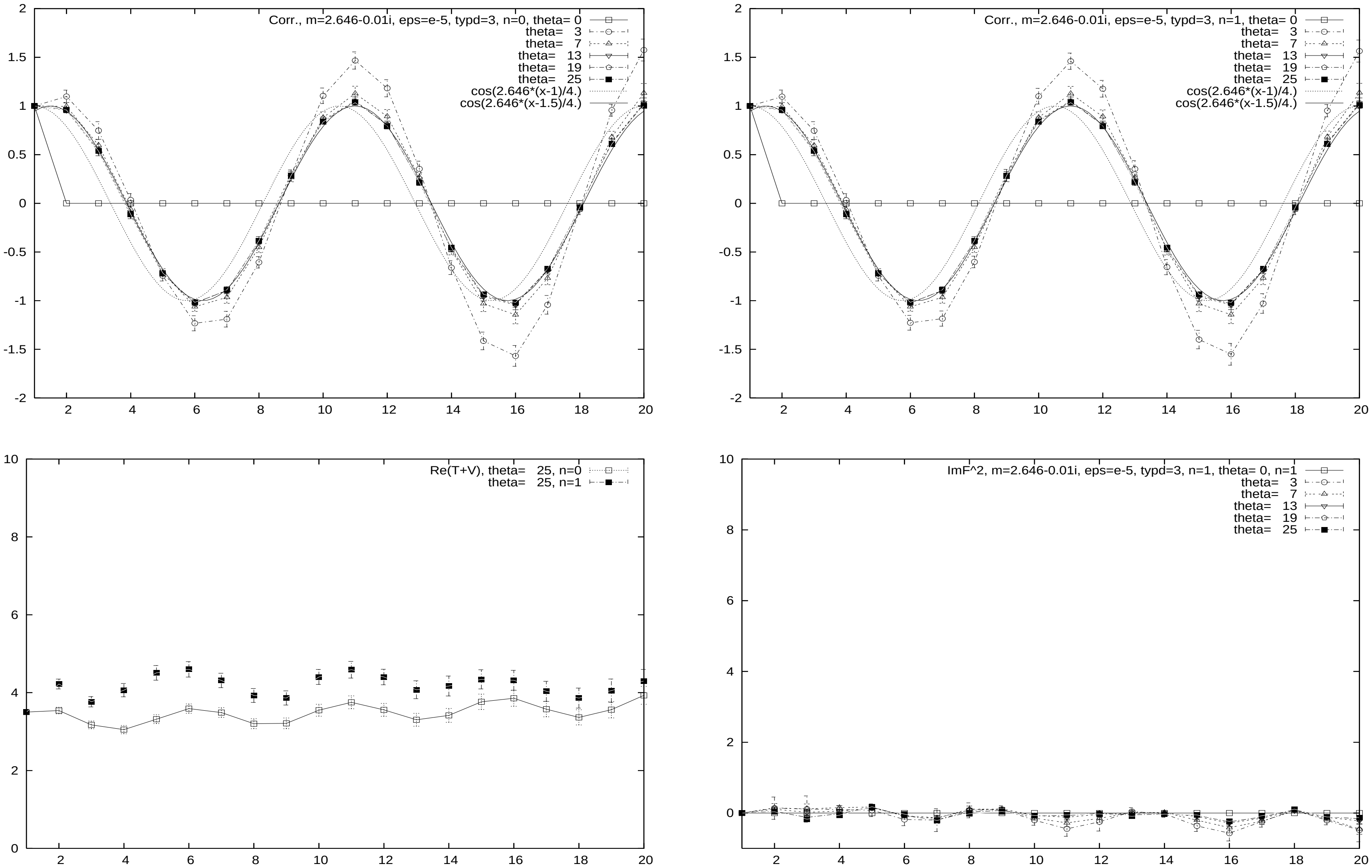


Figure 15: 3+1 dim free field, $12^3.20, \gamma = 4, M = 2.63 - i0.01, \lambda = 0$. Correlations (classical, quantum), energy and $Im < \varphi^2 >$.

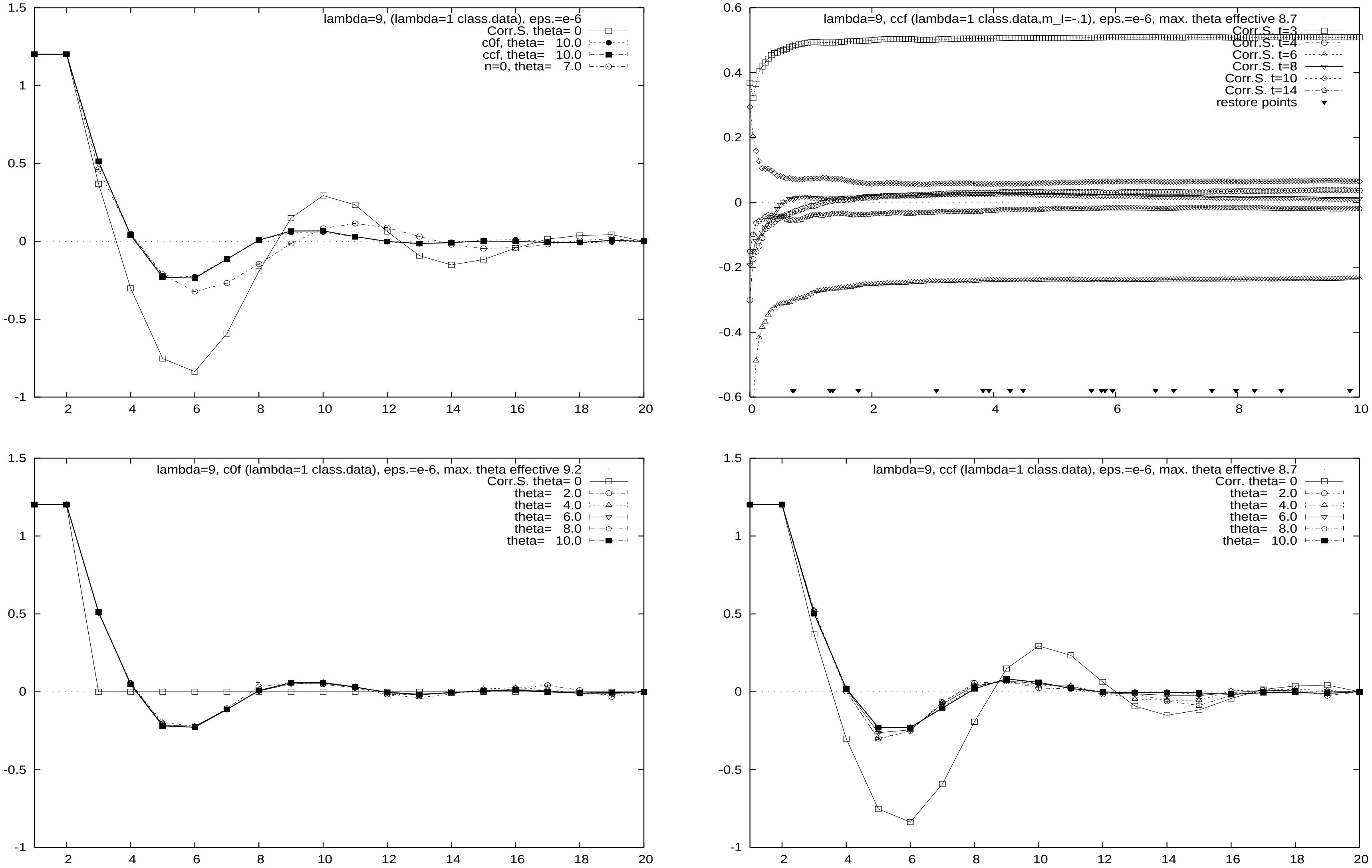
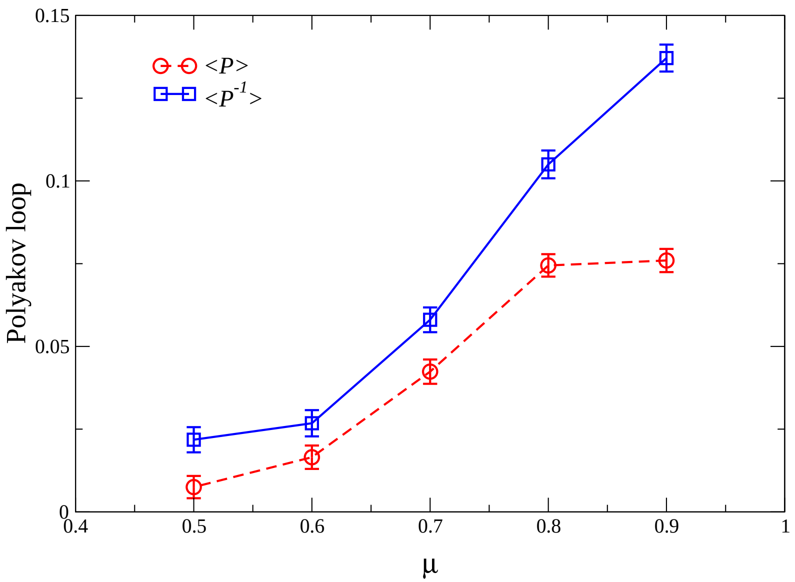


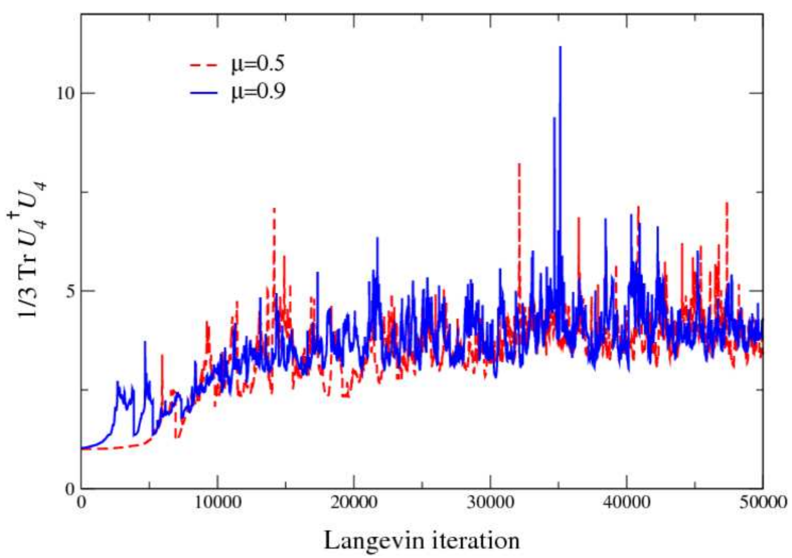
Figure 16: 3+1 dim , φ^4 theory, $8^3.20, M = 0, \lambda = 9, \gamma = 4$. Large ϑ correlations from different starting configurations, ϑ dependence, correlations from different start configuration at various ϑ.

HDM approximation

The structure is similar to the one link $SU(3)$ model, replacing the links by plaquettes in S, etc.

Preliminary, for illustration: the real part of the Polyakov loop and its inverse on a 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$ and 3 flavours *vs* μ, and Tr$U^\dagger U/3$ *vs* θ at $\mu = 0.5$ and 0.9.





7. General observations

1. Run-away trajectories: can be systematically dealt with by adaptive step size. General requirement: *Use high precision since accumulating rounding off errors are very dangereous!*

2. Systematic discretization errors: not relevant at the step size used.

3. Convergence. This is the most urgent problem:

   - For the Minkowski problem one observes some times non-uniqueness of the solution: this appears related to solution multiplicities of the Dyson-Schwinger equations. This may provide a way of approaching this problem.

   - The chemical potential problem appears better behaved, in particular the convergence does not seems to deteriorate with increasing μ (at variance with the MC reweighting method). General tests are, however, missing.

- One can also try to better control the method by redesigning the process. For instance:
  - Kernel controlled LE (to generate positive real parts in the spectrum of the FP Hamiltonian) without changing the averages,
  - reweighting (changing the drift and recalculating the averages),
  - there does not seem to be a tuning problem, but rather definitely wrong choices of parameters or the possibility of systematic improvement.
- We need to design tests as powerfull as the fixed point structure of the classical flow, or the spectrum of the FPE, but which are operational for a lattice problem (many d.o.f.).

β= 1.0	q=	-3.0	q=	-2.0	q=	-1.0	q=	1.0	q=	2.0
p	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
-3.0	-.227E-04	0.114E-02	-.129E-01	0.298E-03	-.176E-02	-.127	-.108	5.87	-22.5	-.997
exact	-.395E-15	0.107E-02	-.128E-01	0.602E-14	0.966E-15	-.127	-.651E-14	5.87	-22.5	0.631E
-2.0	0.924E-04	0.232E-02	-.216E-01	0.858E-03	-.343E-02	-.170	-.109	3.83	-6.67	-.629
exact	0.310E-15	0.217E-02	-.216E-01	-.628E-15	-.135E-15	-.170	-.215E-14	3.83	-6.66	-.341E
-1.0	0.289E-03	0.721E-02	-.450E-01	0.273E-02	-.725E-02	-.259	-.373E-01	1.74	1.21	-.183
exact	0.107E-16	0.563E-02	-.445E-01	-.231E-15	0.339E-15	-.261	0.410E-15	1.74	1.00	-.193E
0.0	20.7	13.9	1.06	-.247	0.146E-01	0.168E-03	0.639E-01	-.350E-03	0.972	-.445
exact	-.372E-16	0.256E-01	-.150	-.729E-16	0.149E-15	-.575	-.138E-15	-.575	-.150	0.363E
1.0	-5.78	-11.1	1.21	0.193	0.435E-01	1.74	0.720E-02	-.257	-.439E-01	-.285E
exact	0.109E-15	-.261	1.00	0.188E-15	-.309E-15	1.74	-.593E-15	-.261	-.445E-01	0.994E
2.0	-.149E+04	-66.6	-6.95	0.382	0.116	3.83	0.350E-02	-.169	-.213E-01	-.857E
exact	-.280E-14	3.83	-6.66	-.135E-14	0.952E-15	3.83	0.185E-15	-.170	-.216E-01	0.355E
3.0	-4.42	-37.2	-22.5	0.974	0.101	5.87	0.183E-02	-.126	-.128E-01	-.296E
exact	-.148E-12	-39.1	-22.5	0.981E-13	0.336E-13	5.87	-.182E-15	-.127	-.128E-01	0.237E
4.0	-8.26	-178.	-46.5	1.33	0.101	7.90	0.112E-02	-.101	-.852E-02	-.112E
exact	-.606E-11	-178.	-46.4	0.163E-11	0.292E-12	7.90	-.402E-13	-.101	-.845E-02	0.196E
5.0	-14.4	-463.	-78.5	1.72	0.100	9.91	0.764E-03	-.837E-01	-.608E-02	-.410E
exact	0.265E-09	-460.	-78.3	-.451E-10	-.569E-11	9.92	-.649E-14	-.838E-01	-.602E-02	-.208E