QCD with fixed complex fermion determinant

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ECT* Trento, 2 March 2009

- **Introduction:** What do we want
  What are we up against

- **New:** Ensembles with $re^{i\theta} = \det(D + \mu\gamma_0 + m)$ fixed

Matter antimatter asymmetry ($\mu \neq 0$)

$$n_q > 0$$

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Here: Fact which we adopt into QCD

Grand canonical approach: *Fix* μ *determine* n_q

$$n_q = \frac{1}{V} \partial_\mu \log Z(\mu)$$

How to include μ in Z

μ is conjugate variable to n_q

$$\mu n_q = \mu \langle q^\dagger q \rangle = \mu \langle \bar{q} \gamma_0 q \rangle$$

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(D_\eta \gamma_\eta + \mu \gamma_0 + m) q + \mathrm{Gluons}$$

Hasenfratz, Karsch, PLB 125 (1983) 308

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μ enters as the 0th component of the gauge field

$$\mathcal{L}_{\mathrm{Lattice\ QCD}} = ... + e^{a\mu}\bar{q}_x\gamma_0 U_{x,x+\hat{0}} q_{x+\hat{0}} + e^{-a\mu}\bar{q}_{x+\hat{0}}\gamma_0 U^\dagger_{x,x+\hat{0}} q_x + ...$$

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Works fi ne for free quarks

Hasenfratz, Karsch, PLB 125 (1983) 308

The sign problem

$$Z_{1+1} = \int dA \, \det{}^2(\underbrace{D}_{\text{Anti Hermitian}} + \underbrace{\mu\gamma_0}_{\text{Hermitian}} + m) \, e^{-S_{\mathrm{YM}}}$$

$$\det{}^2(D + \mu\gamma_0 + m) = |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta}$$

The measure is not real and positive

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Anti Hermitian

Hermitian

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No Monte Carlo sampling of A_η *at* $\mu \neq 0$

What are we up against ?

Drop the phase

$$|\det(D+\mu\gamma_0+m)|^2 = \det(D+\mu\gamma_0+m)\,\det(D-\mu\gamma_0+m)$$

μ becomes an isospin chemical potential

Alford Kapustin Wilczek PRD 59 (1999) 054502

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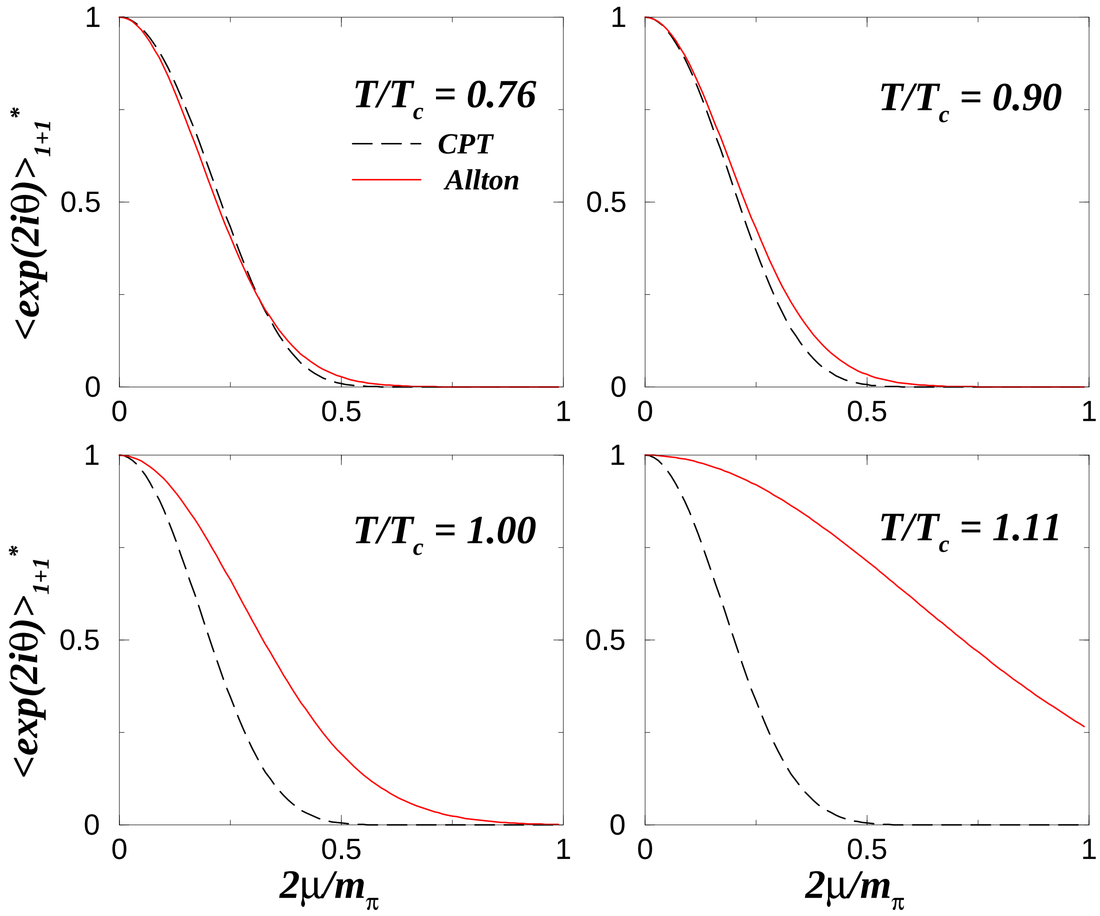
Average phase factor

$$\langle e^{2i\theta}\rangle_{1+1^*} = \frac{Z_{1+1}(\mu_B=\mu)}{Z_{1+1^*}(\mu_I=\mu)} = e^{-V\Delta\Omega}$$

Alford Kapustin Wilczek PRD 59 (1999) 054502

Average phase factor on the lattice

$$\langle e^{2i\theta}\rangle_{1+1^*} = e^{L_i^3 T(c_2 - c_2^I)\mu^2}$$



Allton+... Phys.Rev. D71 (2005) 054508

Splittorff Verbaarschot PRD 77 (2008) 014514

Pions have baryon charge zero

- so how can CPT teach us about μ_B *?*

Certainly

$$\langle n_q \rangle = \frac{1}{V}\partial_\mu \log Z_{CPT} = 0$$

Average phase factor in Chiral Perturbation Theory

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ΔG_0 *is independent of the cutoff*

Splittorff Svetitsky PRD 75 (2007) 114504

Conradi D'Elia PRD 76 (2007) 074501

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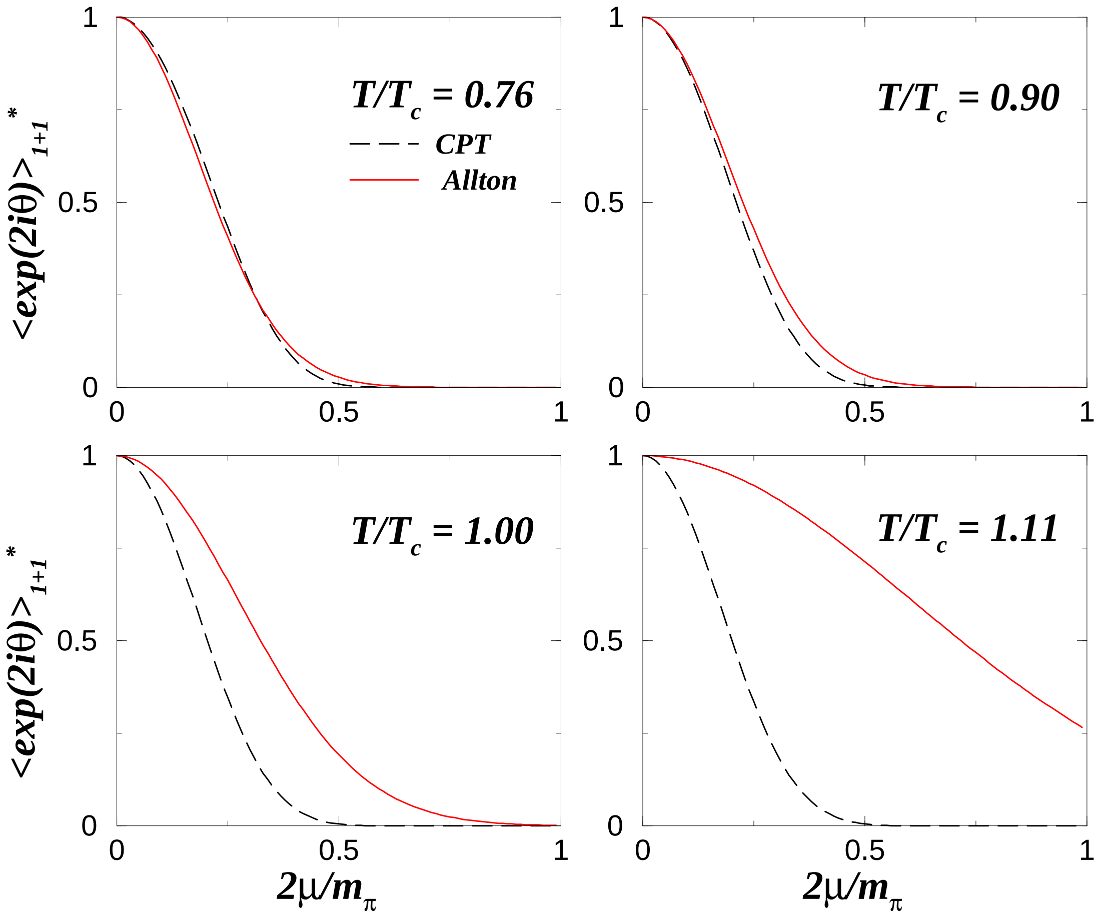

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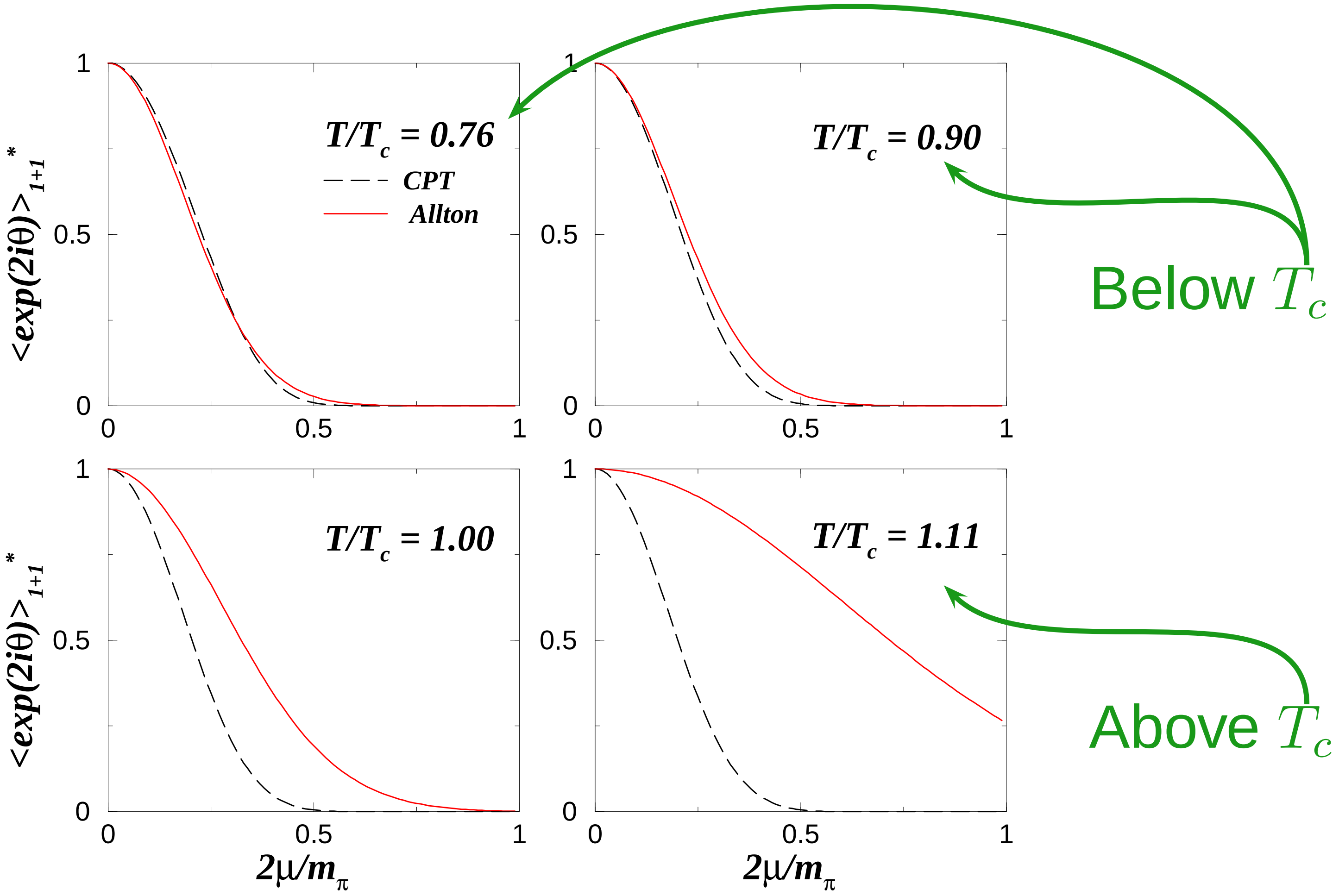

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Counting in CPT (p-regime)

$$m_\pi \sim \mu \sim T \sim \frac{1}{L} \sim \epsilon$$

$$V\Delta\Omega \sim V m_\pi^2 T^2 \sim 1$$

Sign problem $\langle e^{2i\theta} \rangle_{1+1^*} \sim 1$ when $V \to \infty$

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$$\mu < m_\pi/2 \qquad \text{for } T = 0$$

Average phase factor in Chiral Perturbation Theory

Bose condensed phase ($\mu > m_\pi/2$)

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The free energies are different at mean fi eld level

$$\Delta\Omega = 2\mu^2 F^2 + \frac{\Sigma^2 m^2}{2\mu^2 F^2} - 4m\Sigma$$

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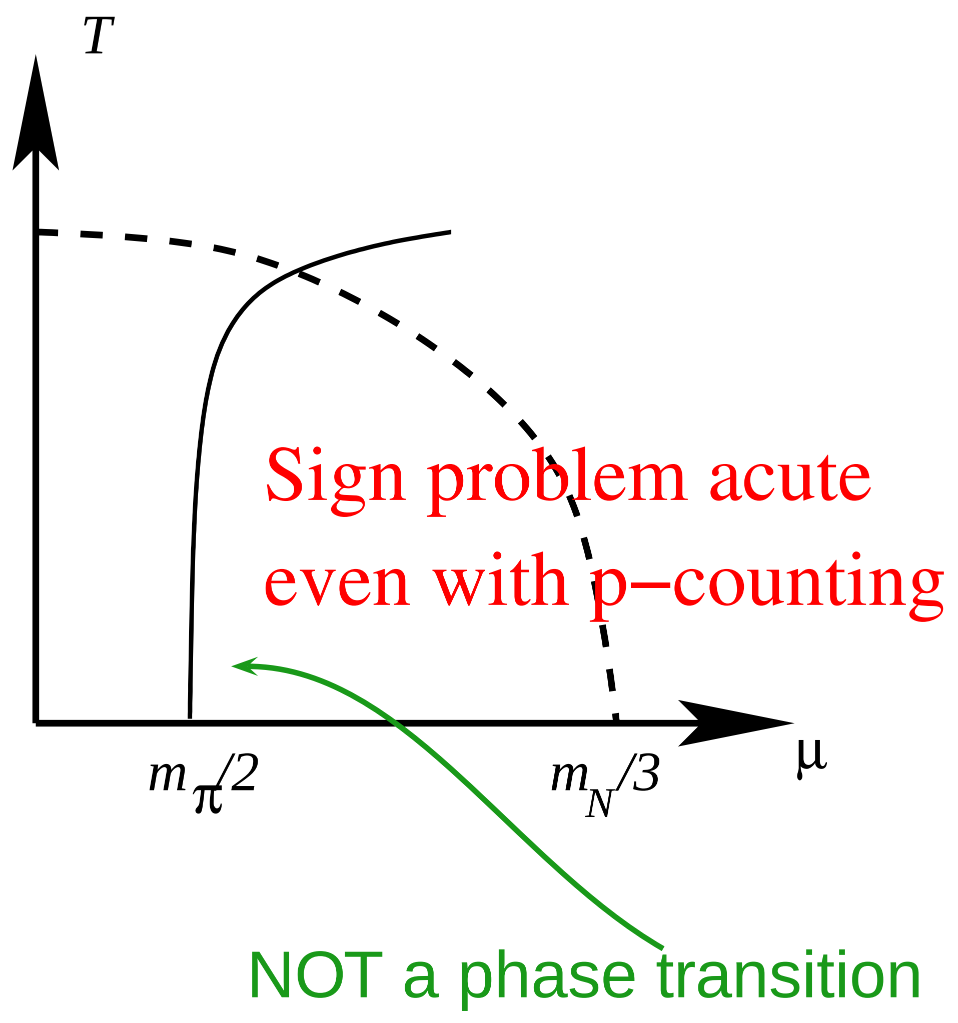
- p-counting $\Rightarrow V\Delta\Omega \sim L^2$

Conclude: $m_\pi/2$ relevant for the sign problem

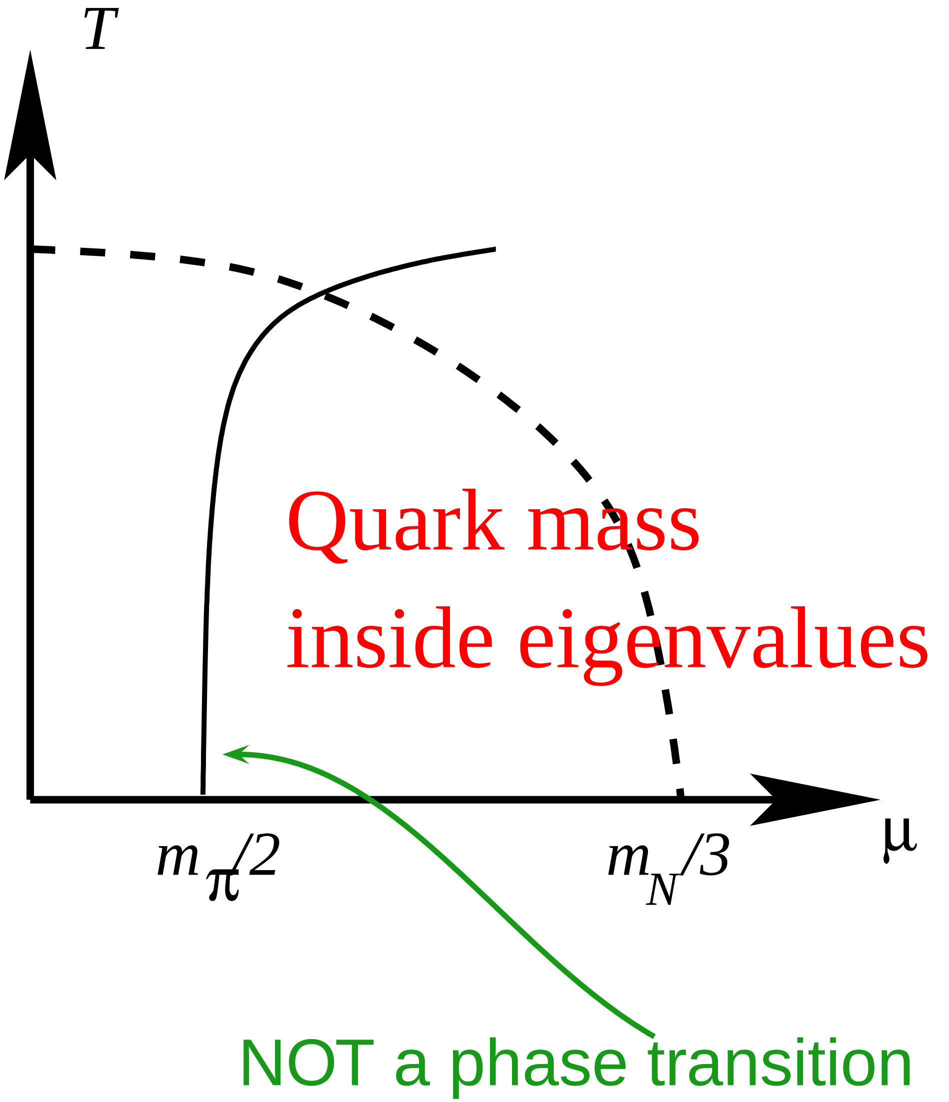


2 final points:

$$Z_{N_f=2} = \int dA\, \det{}^2(D + \mu\gamma_0 + m)\; e^{-S_{\rm YM}}$$

Anti Hermitian

Hermitian



Gibbs PRINT-86-0389

Bosonic quarks = average inverse determinants

$$Z_{N_f=-1} = \left\langle \frac{1}{\det(D + \mu\gamma_0 + m)} \right\rangle$$

Bergère arXiv:hep-th/0404126

Akemann, Pottier, J.Phys. A37 (2004) 453

Splittorff, Verbaarschot Nucl.Phys. B757 (2006) 259

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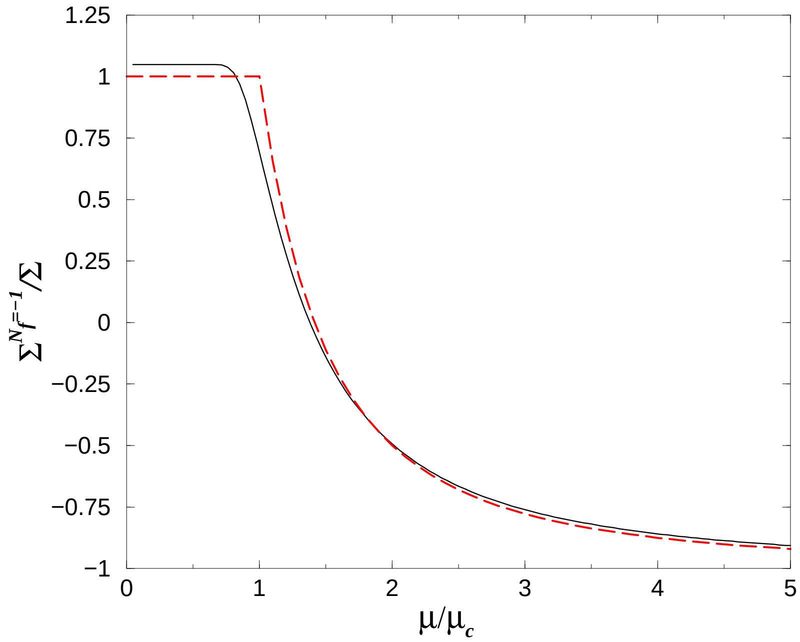
SURPRISE: phase transition at $\mu = m_\pi/2$



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In σ-model: Because of convergence requirements

$$Z_{N_f=-1} = \left\langle \frac{\det(D + \mu\gamma_0 + m)^*}{\det \begin{pmatrix} \epsilon & D + \mu\gamma_0 + m \\ (D + \mu\gamma_0 + m)^* & \epsilon \end{pmatrix}} \right\rangle$$

Feinberg Zee NPB 504 (1997) 578
Splittorff, Verbaarschot NPB 757 (2006) 259
Splittorff, Verbaarschot, Zirnbauer NPB 803 (2008) 381

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At Mean Field level in σ-model:

$$\left\langle \frac{1}{\det} \right\rangle \sim \frac{\langle \det^* \rangle}{\langle \det \det^* \rangle}$$

Feinberg Zee NPB 504 (1997) 578

Splittorff, Verbaarschot NPB 757 (2006) 259

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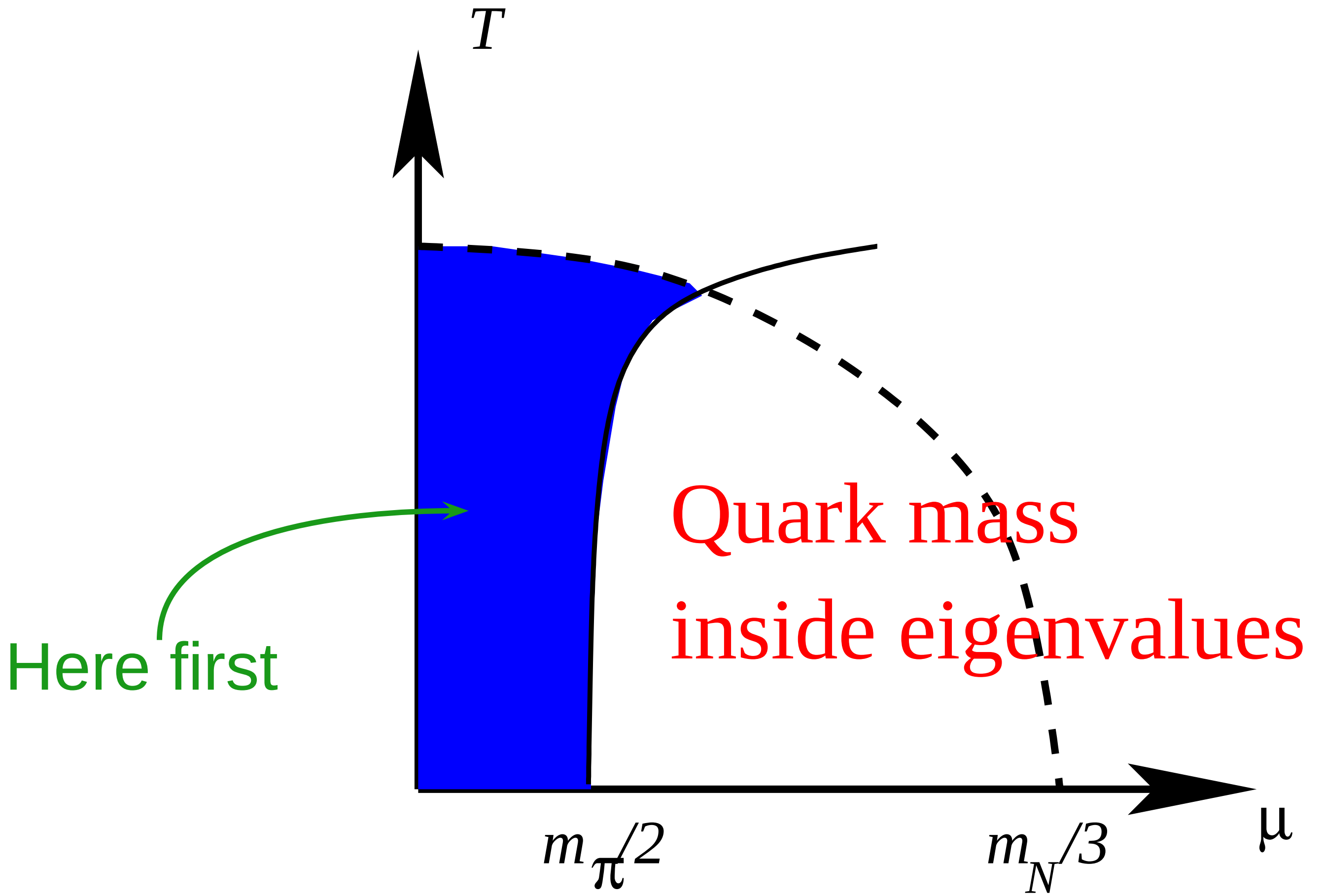
End Intro

- **What:** Ensembles with $re^{i\theta} = \det(D + \mu\gamma_0 + m)$ fixed
- **Why:** Understand how Z, n_q, $\langle\bar{\psi}\psi\rangle$ build up as $\int drd\theta$
- **How:** Analytically in Chiral Perturbation Theory

Fixing θ



T
Here first
Quark mass
inside eigenvalues
$m_\pi/2$
$m_N/3$
μ

The delta function & the moments

$$\langle \delta(2\theta - 2\theta') \rangle_{1+1} \equiv \frac{1}{Z_{1+1}} \int dA\, \delta(2\theta - 2\theta') \mathrm{det}^2(D+\mu\gamma_0+m) e^{-S_{YM}}$$

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The moments of the phase factor

$$\langle e^{2ip\theta'} \rangle_{N_f} \equiv \left\langle \frac{\det^{p}(D + \mu\gamma_0 + m)}{\det^{p}(D - \mu\gamma_0 + m)} \right\rangle_{N_f}$$

is a ratio of two partition functions

Phase transition at $\mu = m_\pi/2$

$$\langle e^{2ip\theta'} \rangle_{N_f} = \frac{Z_{N_f+p|p^*}}{Z_{N_f}} = e^{-V\Delta\Omega_p}$$

In CPT at 1-loop

Number of charged pions

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ΔG_0 is the difference between charged and neutral pions

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The θ-distribution from CPT

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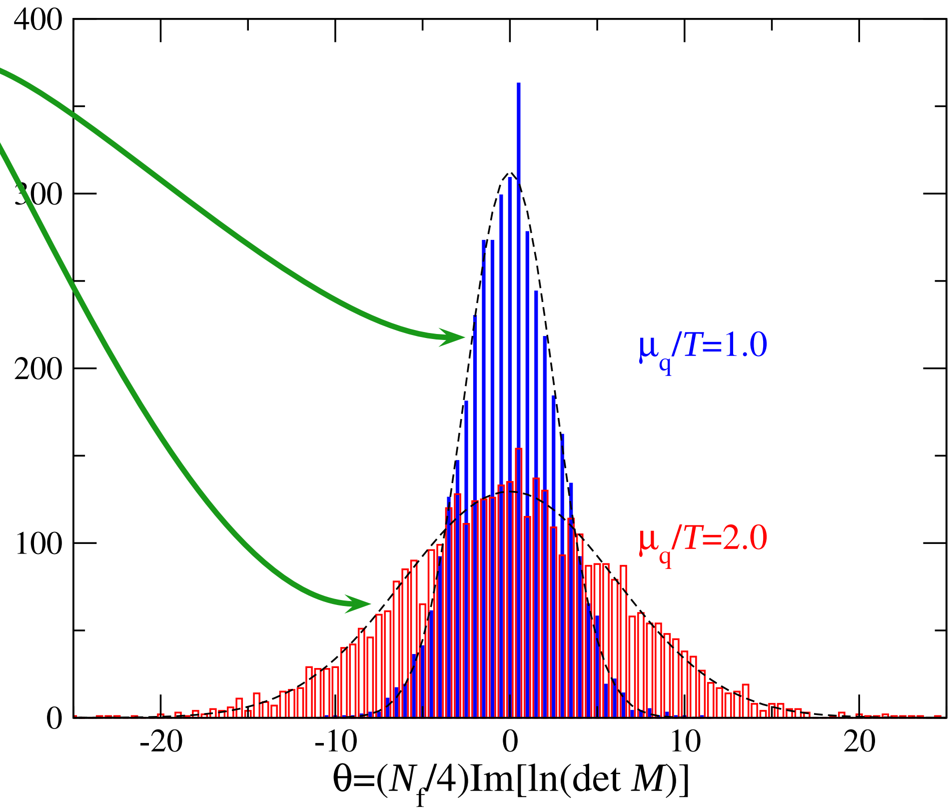
Prediction: Gaussian folded onto $[-\pi : \pi] \times$ phase

$$\langle \delta(2\theta - 2\theta') \rangle_{1+1} = \frac{e^{V\Delta G_0}}{\sqrt{\pi V \Delta G_0}} e^{2i\theta} \sum_{n=-\infty}^{\infty} e^{-(\theta+2\pi n)^2/V\Delta G_0}$$

The θ-distribution from the lattice

Gaussian fi ts



CPT $\Rightarrow$ width $\sqrt{V\Delta G_0}$

Ejiri PRD 77 (2008) 014508

The distribution of n_q with θ

$$\begin{aligned}
&\langle n_q \delta(\theta - \theta')\rangle_{1+1} \\
\equiv \quad & \frac{1}{Z_{1+1}} \lim_{\tilde{\mu}\to\mu} \frac{d}{d\tilde{\mu}} \int dA\, \delta(\theta - \theta'(\mu)) \det{}^2(D + \tilde{\mu}\gamma_0 + m) e^{-S_{YM}}
\end{aligned}$$

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For $\theta \in [-\infty, \infty]$

$$\delta(\theta - \theta'(\mu)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\; e^{-ip\theta} \frac{\mathrm{det}^{p/2}(D + \mu\gamma_0 + m)}{\mathrm{det}^{p/2}(D - \mu\gamma_0 + m)}$$

The distribution of n_q with θ from CPT

$$\langle n_q \delta(\theta - \theta') \rangle_{1+1}$$

$$= \quad [\lim_{\tilde{\mu}\to\mu} \frac{d}{d\tilde{\mu}} \Delta G_0(-\mu, \tilde{\mu})](1 + i\frac{\theta}{\Delta G_0})\frac{e^{\Delta G_0}}{\sqrt{\pi \Delta G_0}} e^{2i\theta} e^{-\theta^2/\Delta G_0}$$

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Certainly

$$\langle n_q \rangle_{1+1} = \int_{-\infty}^{\infty} d\theta \; \langle n_q \delta(\theta - \theta') \rangle_{1+1} = 0$$

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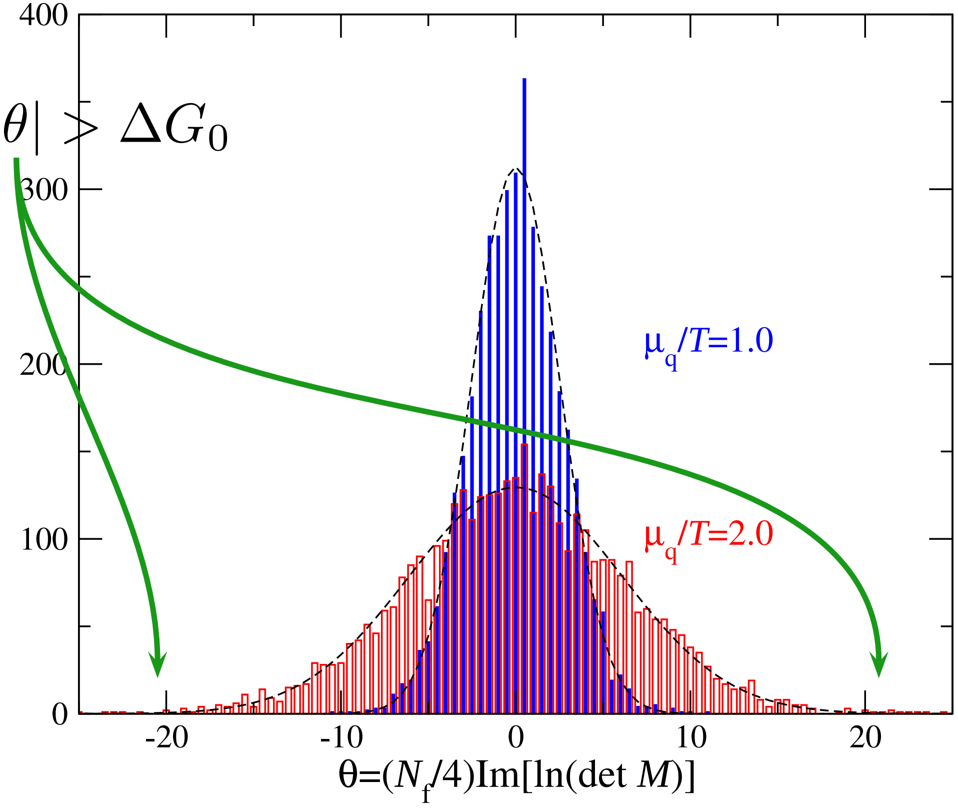
Integration range: $\theta_{max} \gg \Delta G_0$ to get $\langle n_q \rangle_{1+1} \to 0$

The θ-distribution from the lattice

n_q sensitive to $|\theta| \gtrsim \Delta G_0$



CPT $\Rightarrow$ width $\sqrt{\Delta G_0}$

Ejiri PRD 77 (2008) 014508

The distribution of $\bar{\psi}\psi$ with θ from CPT

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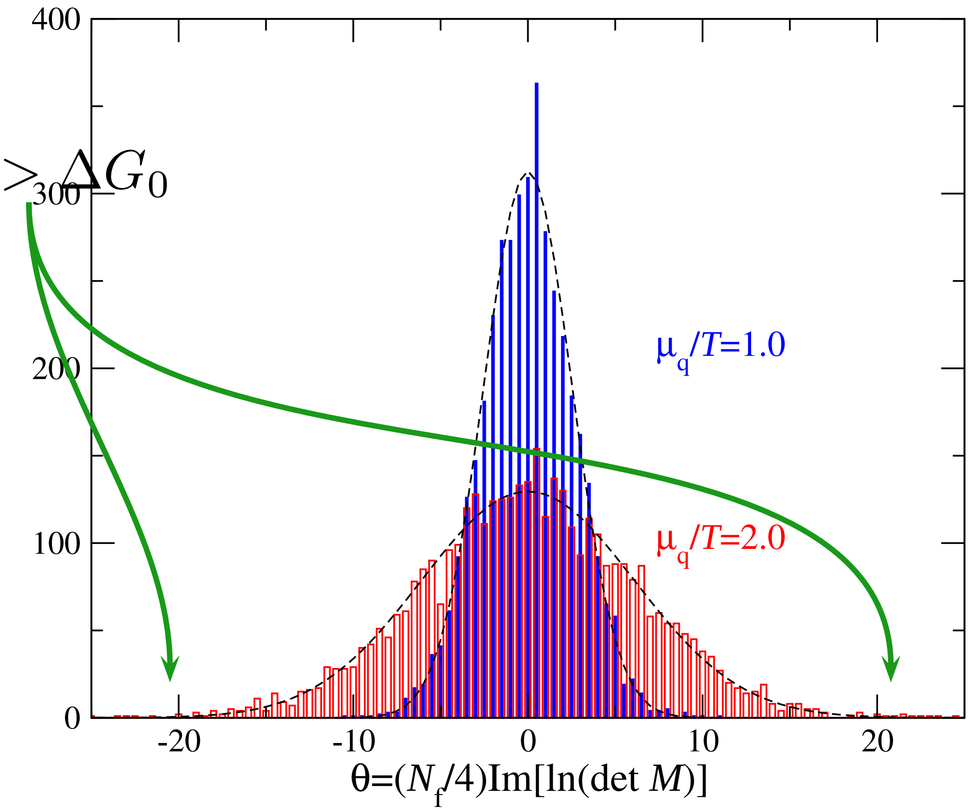
The θ-distribution from the lattice

n_q & $\langle\bar{\psi}\psi\rangle$

sensitive to $|\theta| > \Delta G_0$



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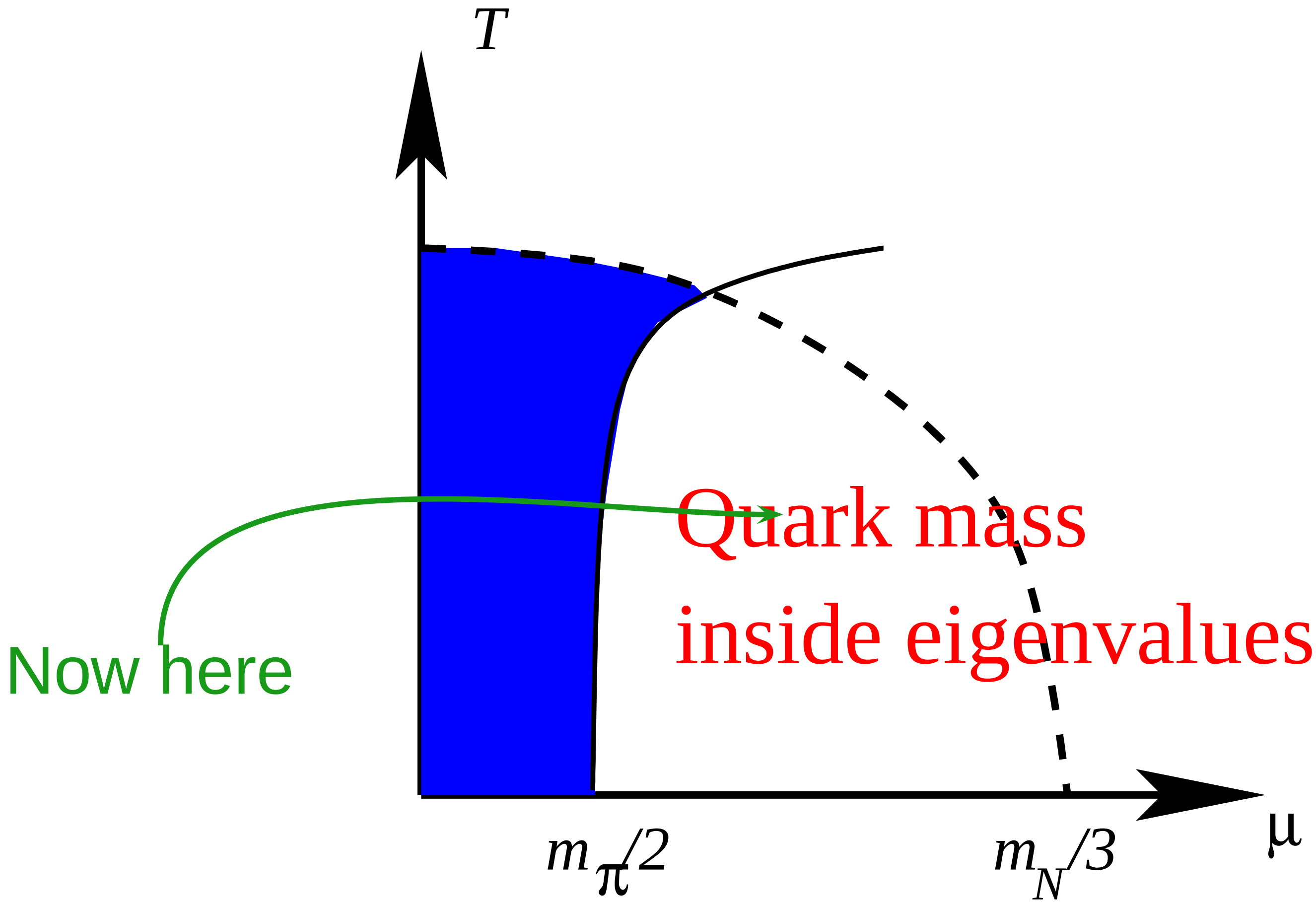

T
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inside eigenvalues
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Bosonic mean fi eld rules

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$$\Delta\Omega = 2\mu^2 F^2 + \frac{\Sigma^2 m^2}{2\mu^2 F^2} - 4m\Sigma$$

Lombardo Splittorff Verbaarschot

The θ-distribution ($\mu > m_\pi/2$)

$$\langle \delta(2\theta - 2\theta') \rangle = \frac{1}{\pi} \sum_{p=-\infty}^{\infty} e^{-2ip\theta} \langle e^{2ip\theta'} \rangle_{N_f}$$

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Lorentzian (on $[-\pi : \pi]$)

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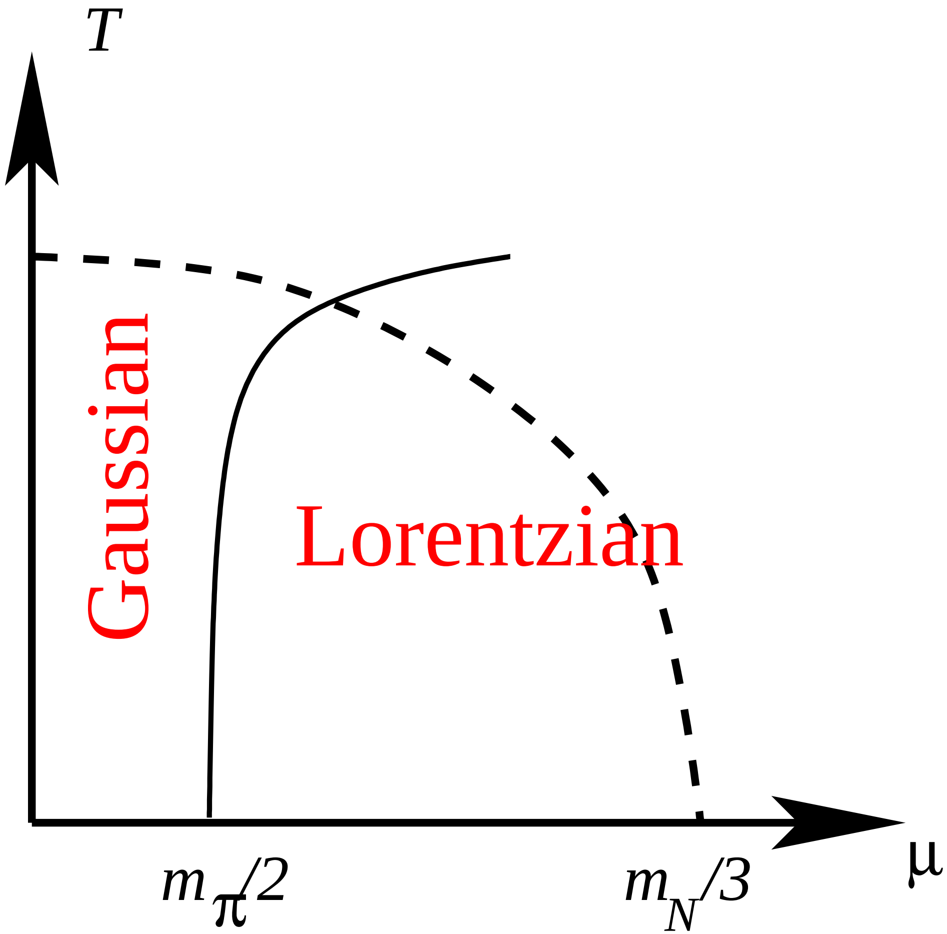
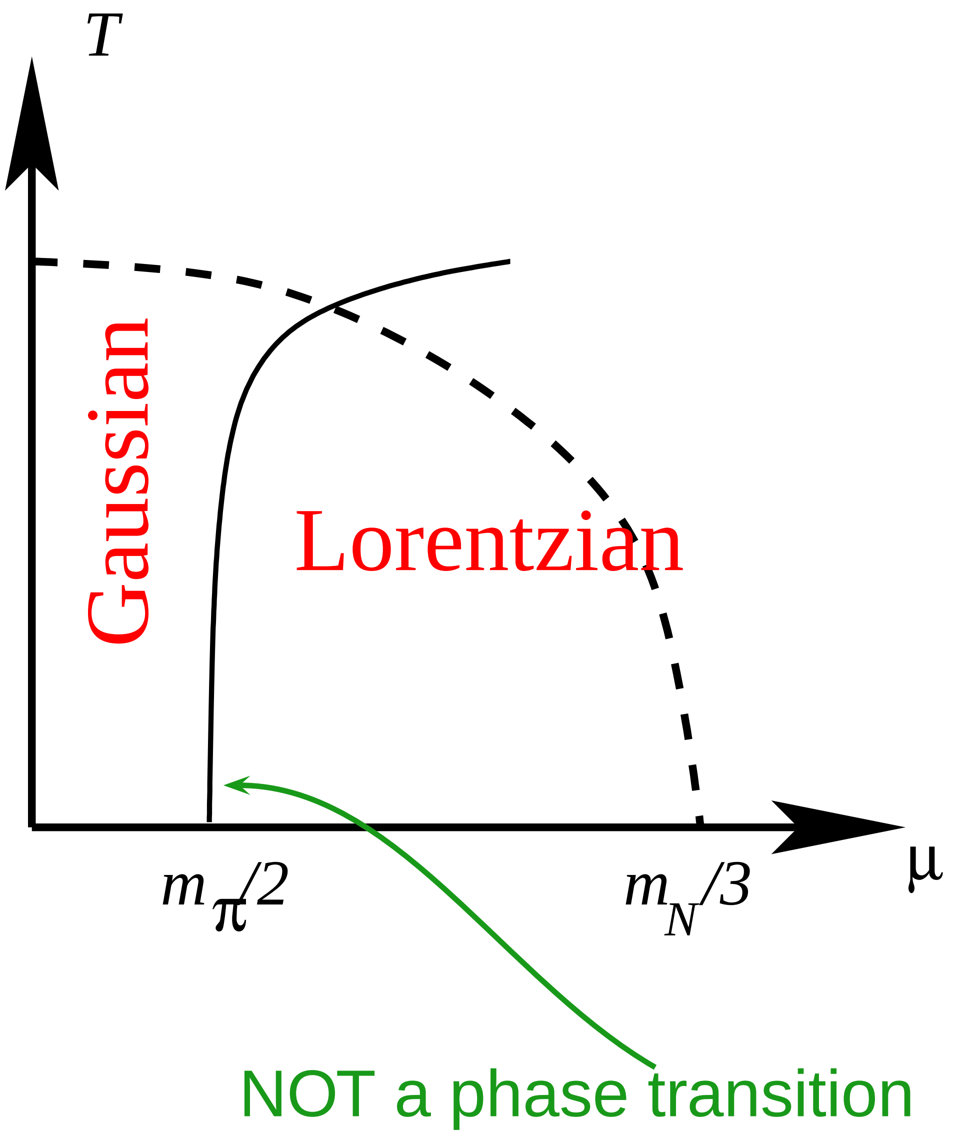
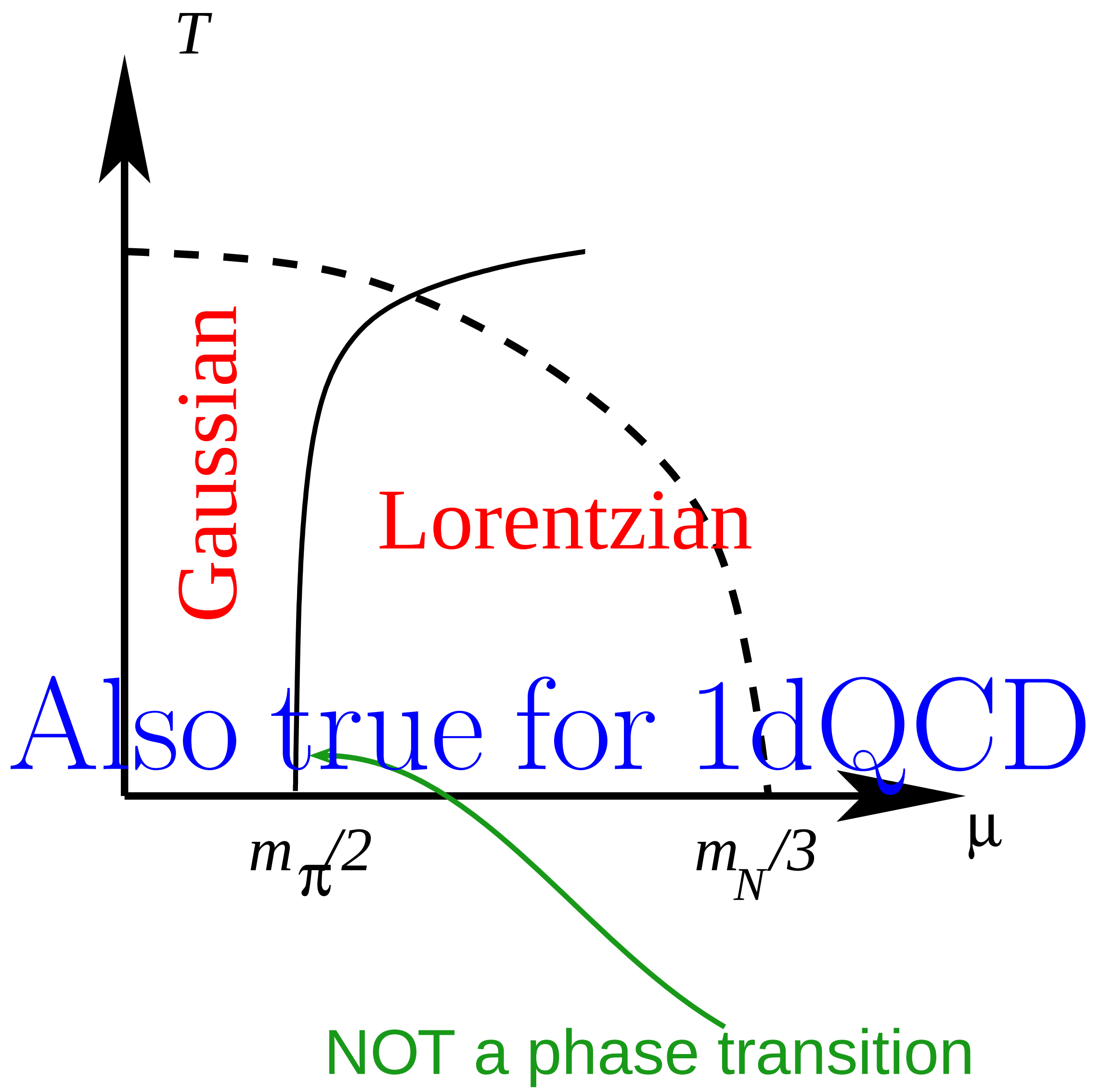
$$\text{Central limit theorem fails!}$$

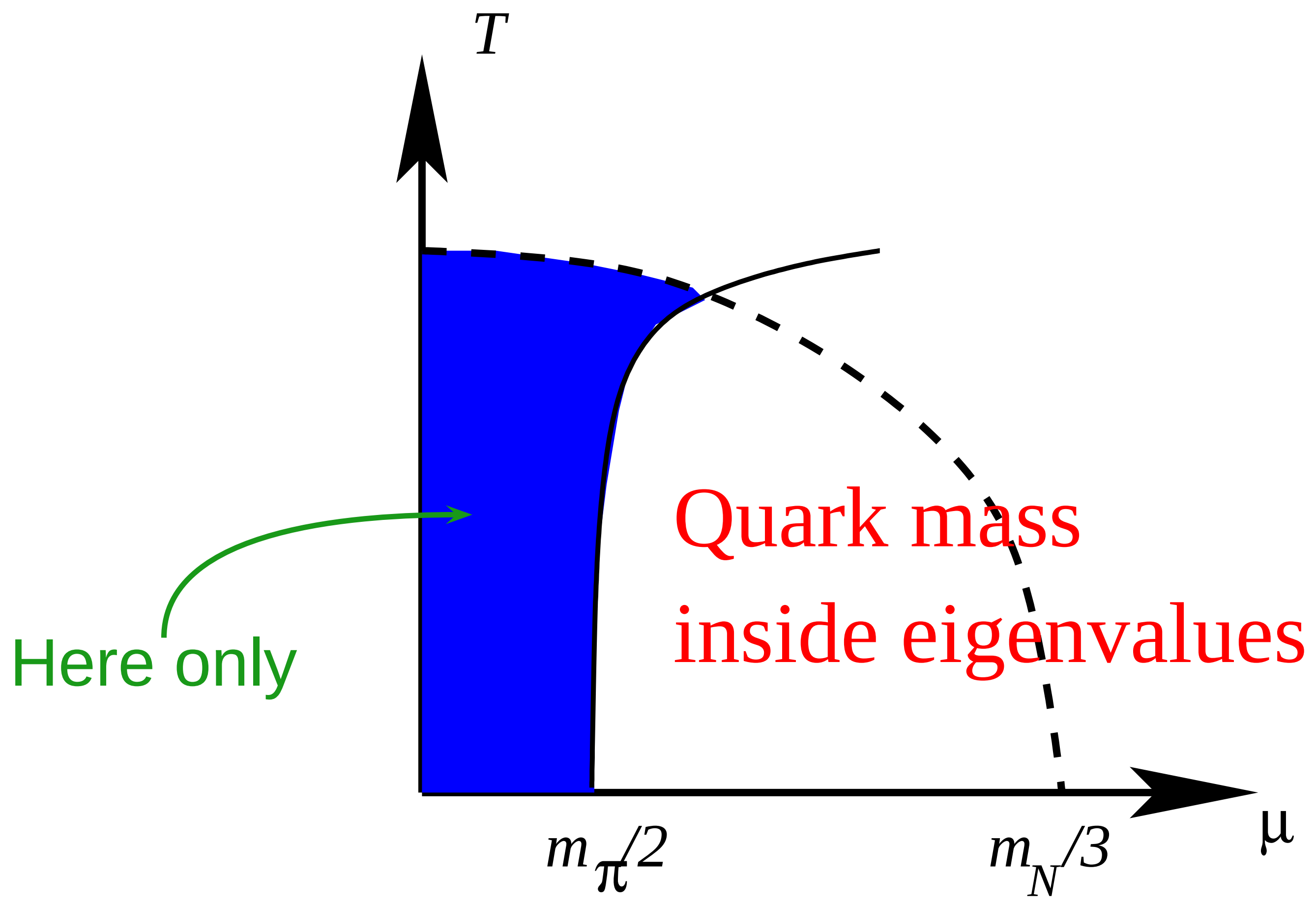

T
Gaussian
Lorentzian
μ
$m_\pi/2$
$m_N/3$


T
Gaussian
Lorentzian
μ
mπ/2
mN/3
NOT a phase transition


T
Gaussian
Lorentzian
Also true for 1dQCD
μ
$m_\pi/2$
$m_N/3$
NOT a phase transition

Fixing $|\det(D + \mu\gamma_0 + m)|$

Fixing $|F| \equiv \frac{|\det(D+\mu\gamma_0+m)|}{\det(D+m)}$


T
Quark mass
inside eigenvalues
Here only
m_π/2
m_N/3
μ

The $|F|$ distribution

$$|F|' \equiv \frac{|\det(D + \mu\gamma_0 + m)|}{\det(D + m)}$$

The δ-function

$$\langle \delta(|F| - |F|') \rangle_{1+1} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} |F|^{ip} \langle (|F|')^{-ip-1} \rangle_{1+1}$$

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Compute all moments of $|F|$ in CPT

The $|F| = |\det(D+\mu\gamma_0+m)|/\det(D+m)|$ distribution in CPT

$\mu < m_\pi/2$

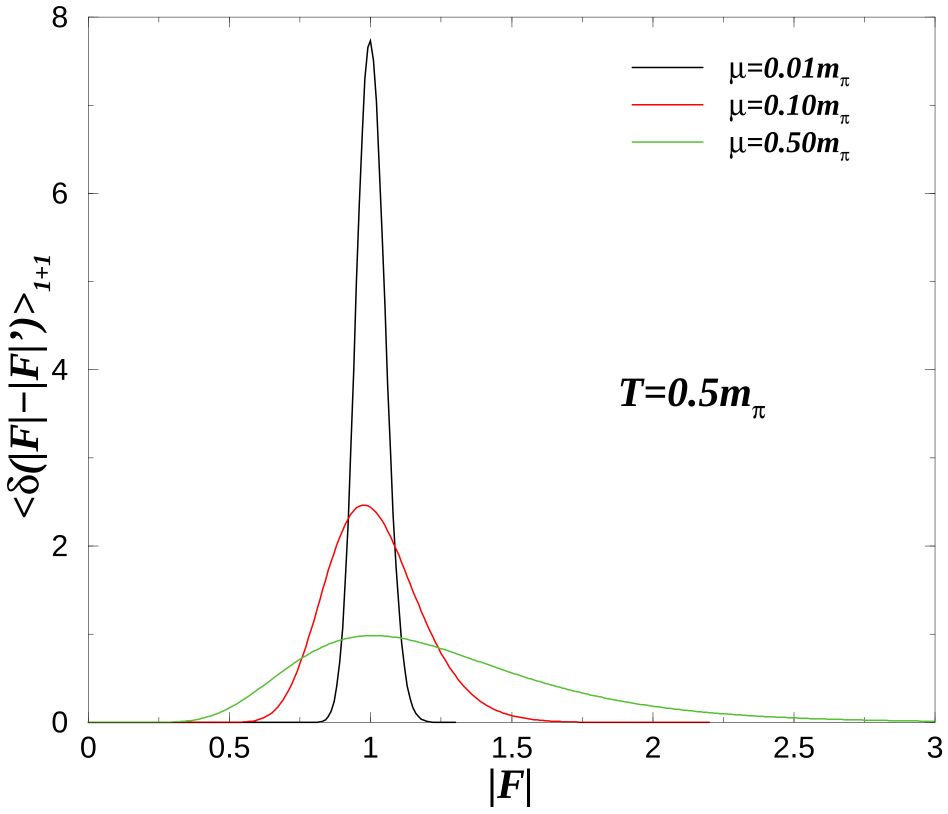
$$\langle\delta(|F|-|F|')\rangle_{1+1} = \frac{\exp[-\frac{A^2}{B}]}{\sqrt{\pi B}}\ |F|^{2\frac{A}{B}-1}\ e^{-\frac{\log^2|F|}{B}}$$

Where

$$\begin{aligned} A &= V\Delta G_0(\mu) - 2V\Delta G_0(\frac{\mu}{2}) \\ B &= V\Delta G_0(\mu) - 4V\Delta G_0(\frac{\mu}{2}) \end{aligned}$$

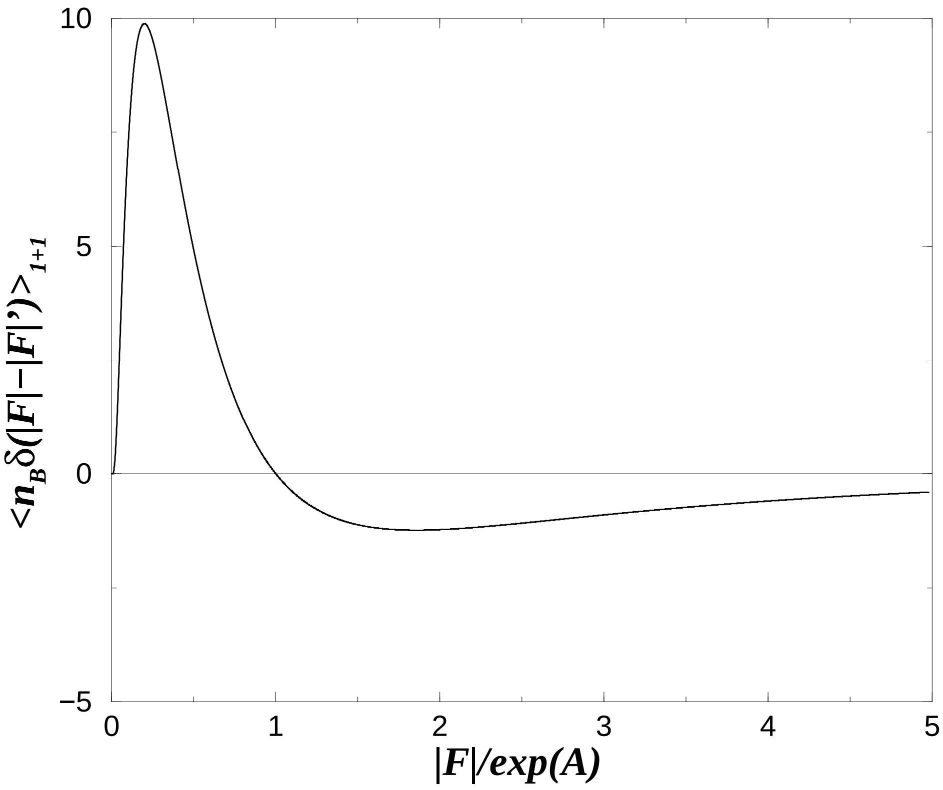
The $|F| = |\det(D + \mu\gamma_0 + m)|/\det(D + m)|$ distribution in CPT

$\mu < m_\pi/2$



The distribution of n_q with $|F|$ $\quad (A \sim V)$

$$\langle n_q \delta(|F| - |F|') \rangle_{1+1} \sim (A - \log |F|) \langle \delta(|F| - |F|') \rangle_{1+1}$$



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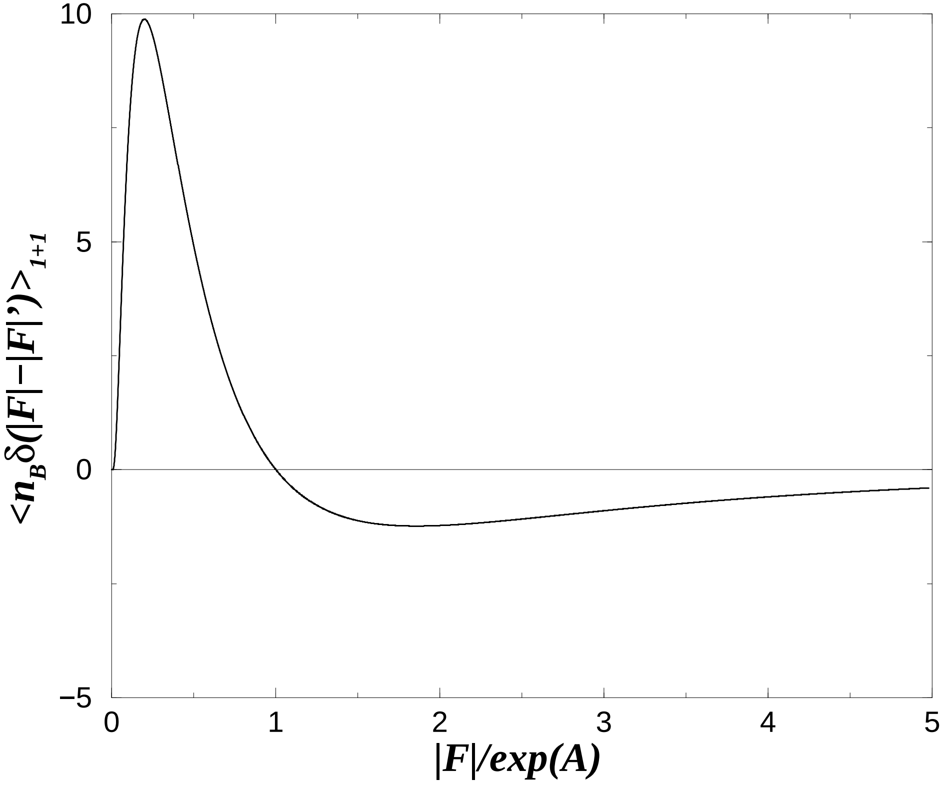


Certainly $\langle n_q \rangle_{1+1} = 0$

Fixing θ and $|F| \equiv \frac{|\det(D+\mu\gamma_0+m)|}{\det(D+m)}$

1-loop CPT $\mu < m_\pi/2$

No correlations between $|F|$ *and* $e^{ip\theta'}$ *from* π*'s*

1-loop CPT $\mu < m_\pi/2$

No correlations between $|F|$ *and* $e^{ip\theta'}$ *from* π*'s*

$$\langle \delta(|F| - |F|')\delta(\theta - \theta')\rangle_{1+1} = \langle \delta(|F| - |F|')\rangle_{1+1}\langle \delta(\theta - \theta')\rangle_{1+1}$$

1-loop CPT $\mu < m_\pi/2$

No correlations between $|F|$ *and* $e^{ip\theta'}$ *from* π*'s*

$$\begin{aligned}
&\langle n_q \delta(|F| - |F|')\delta(\theta - \theta')\rangle_{1+1} \\
&\qquad = \langle n_q \delta(|F| - |F|')\rangle_{1+1}\langle \delta(\theta - \theta')\rangle_{1+1} \\
&\qquad\quad + \langle \delta(|F| - |F|')\rangle_{1+1}\langle n_q \delta(\theta - \theta')\rangle_{1+1}
\end{aligned}$$

1-loop CPT $\mu < m_\pi/2$

No correlations between $|F|$ *and* $e^{ip\theta'}$ *from* π*'s*

$$\begin{aligned} &\langle n_q \delta(|F| - |F|')\delta(\theta - \theta')\rangle_{1+1} \\ &\qquad = \langle n_q \delta(|F| - |F|')\rangle_{1+1} \langle \delta(\theta - \theta')\rangle_{1+1} \\ &\qquad\quad + \langle \delta(|F| - |F|')\rangle_{1+1} \langle n_q \delta(\theta - \theta')\rangle_{1+1} \end{aligned}$$

n_q *is sensitive to* $\theta \sim V$ *and* $\log|F| \sim V$

Conclusions

Interplay between lattice QCD and analytic QCD is essential to understand QCD at $\mu \neq 0$

Conclusions

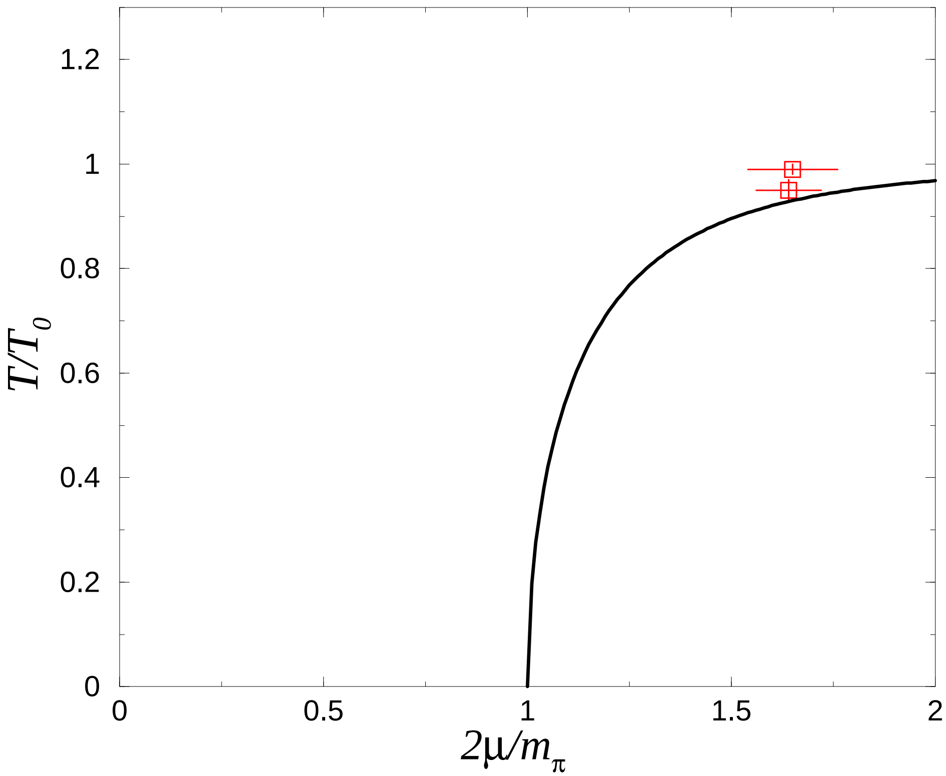
Interplay between lattice QCD and analytic QCD is essential to understand QCD at $\mu \neq 0$

Here:

- Fixed θ and $|F|$
- Studied how $\langle n_q \rangle$ becomes zero
- Studied cancellations leading to $\langle \bar{\psi}\psi \rangle$
- Directly linked to method of Ejiri
- Break down of central limit theorem (Lorentzian)

Additional slides

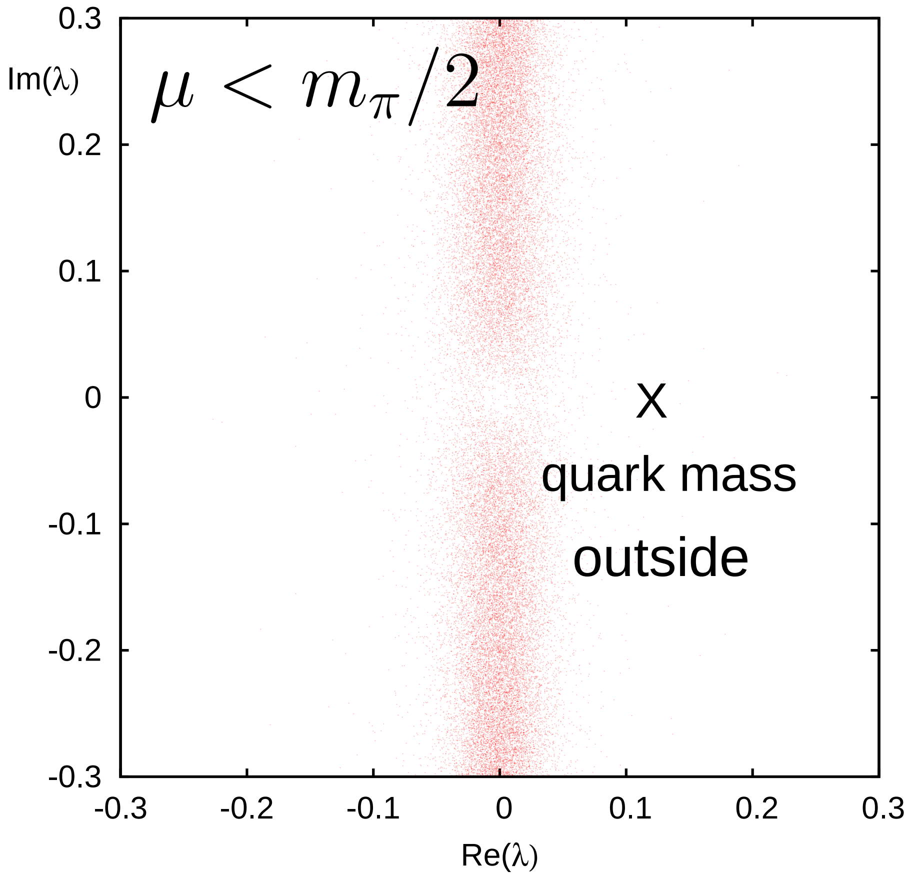
Lattice measurement of endpoint

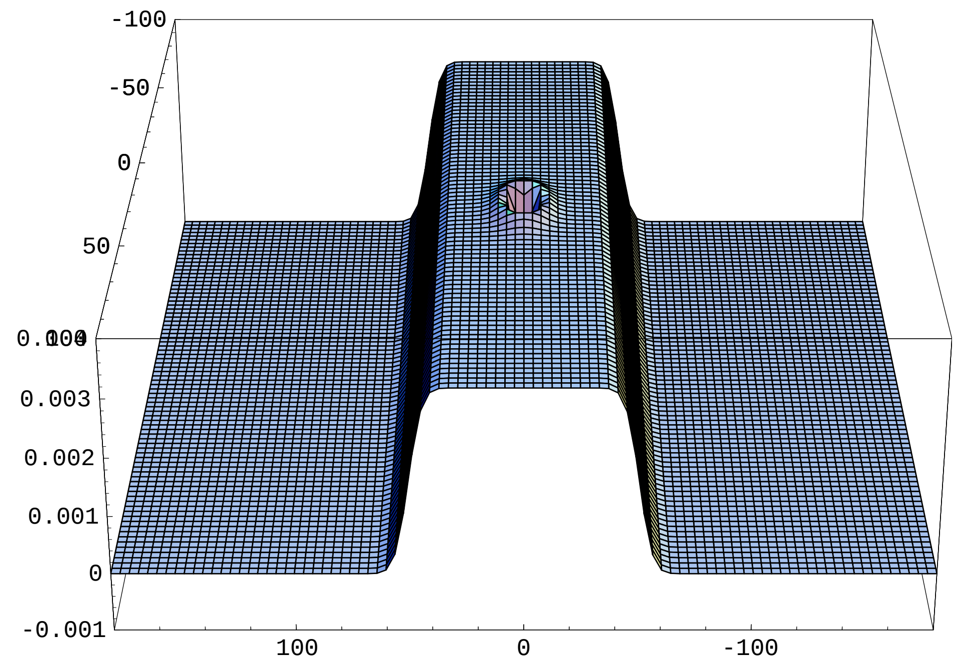


Fodor Katz JHEP 0203:014,2002; JHEP 0404:050,2004
Splittorff, hep-lat/0505001, PoS LAT2006:023,2006

Philipsen 0710.1217

Quark mass & the eigenvalue distribution





$$(D + \mu\gamma_0)\psi_k = z_k\psi_k$$

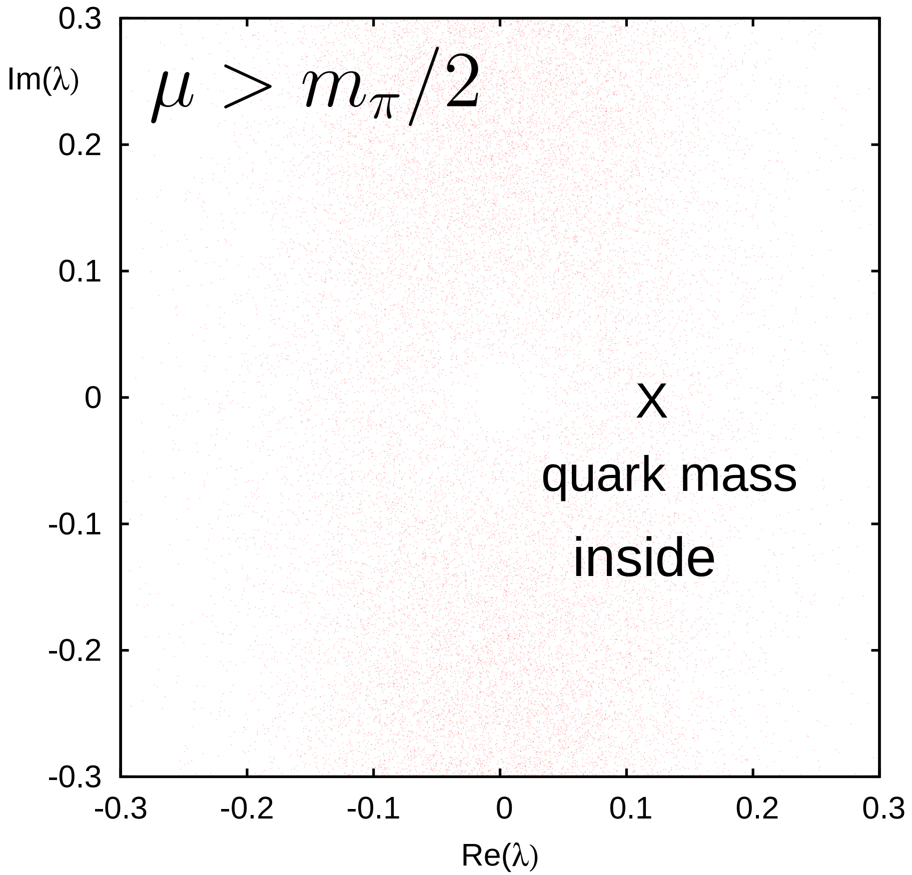
$$\det(D + \mu\gamma_0 + m) = \prod_k z_k + m$$

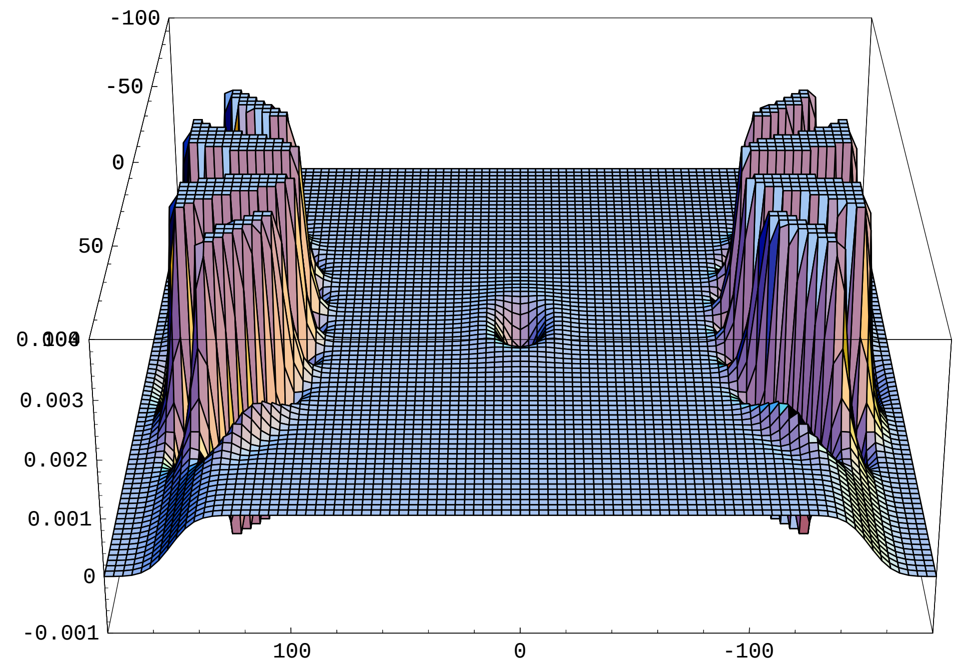
Bloch Wettig Lattice 2006

Gibbs PRINT-86-0389

Splittorff, Verbaarschot NPB 683 (2004) 467

Quark mass & the eigenvalue distribution





For $2\mu > m_\pi$ the density is complex and oscillates

Bloch Wettig Lattice 2006

Osborn PRL 93 (2004) 222001

Akemann Osborn Splittorff Verbaarschot NPB 712 (2005) 287

The chiral condensate from the eigenvalue density

$$\begin{aligned}\langle\bar{\psi}\psi\rangle(m) &= \frac{1}{V}\partial_m \log Z(m;\mu) \\ &= \frac{1}{V}\int dxdy\,\rho(x,y)\frac{1}{x+iy+m}\end{aligned}$$

The oscillations of the density are responsible for chiral symmetry breaking

Osborn Splittorff Verbaarschot PRL 94 (2005) 202001

Ravagli Verbaarschot arXiv:0704.1111