Recent Results of Stochastic Quantisation

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Complex Langevin method and real time evolution

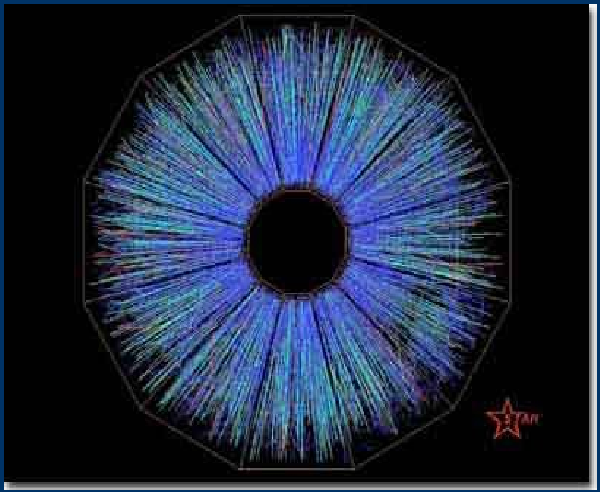
Results for a scalar oscillator, SU(2) gauge theory

Connection with Schwinger Dyson equations

Optimization methods: reweighting, gauge fixing

Motivations

Real-time evolution



High occupation numbers
At $n = O(1/\alpha)$ all diagrams become large
preventing perturbative treatment

On the lattice: mainly equilibrium methods so far, static quantities
with few exceptions

Late times approaching thermal equilibrium:
Classical approximation breaks down

Direct Method: Schödinger equation for the wave function: $\Psi[A_\mu^a(x)]$
Impossible!

Formulation with non-equilibrium generator function $Z[J] = \int D\Phi \, e^{i \int_C L(\Phi, J) dt}$

averages with complex weight is needed! $e^{i S_M}$
Importance sampling doesn't work

Stochastic Quantization

Parisi, Wu (1981)

Weighted, normalized average: $\langle O\rangle = \dfrac{\int e^{-S(x)} O(x)\,dx}{\int e^{-S(x)}\,dx}$

Stochastic process for x: $\dfrac{dx}{d\tau} = -\dfrac{\partial S}{\partial x} + \eta(\tau)$

Gaussian noise $\langle \eta(\tau)\rangle = 0 \quad \langle \eta(\tau)\eta(\tau')\rangle = 2\delta(\tau-\tau')$

Averages are calculated along the trajectories:

$$\langle O\rangle = \frac{1}{T}\int_0^T O(x(\tau))\,d\tau$$

Fokker-Planck equation for the probability distribution of P(x):

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} + P\frac{\partial S}{\partial x}\right) = -H_{FP}P$$

Real action $\rightarrow$ positive eigenvalues

for real action the Langevin method is convergent

Real-time evolution

$\langle O(t)\rangle=\langle i|U(0,t)OU(t,0)|i\rangle$ Schwinger-Keldysh contour

Nonequilibrium generating functional $Z[J]=\int D\Phi\, e^{i\int_c L(\Phi,J)dt}$

Real time= Langevin method with complex action! $\frac{d\phi}{d\tau}=i\frac{\partial S}{\partial \phi}+\eta(\tau)$

Klauder '83, Parisi '83, Hueffel, Rumpf '83,
Okano, Schuelke, Zeng '91, ...
applied to nonequilibrium: Berges, Stamatescu '05, ...

5D classical langevin system $\longrightarrow$ 4D quantum averages

The field is complexified

real scalar $\longrightarrow$ complex scalar

link variables: SU(2) $\longrightarrow$ SL(2,C)
compact non-compact

Is it still the same theory?

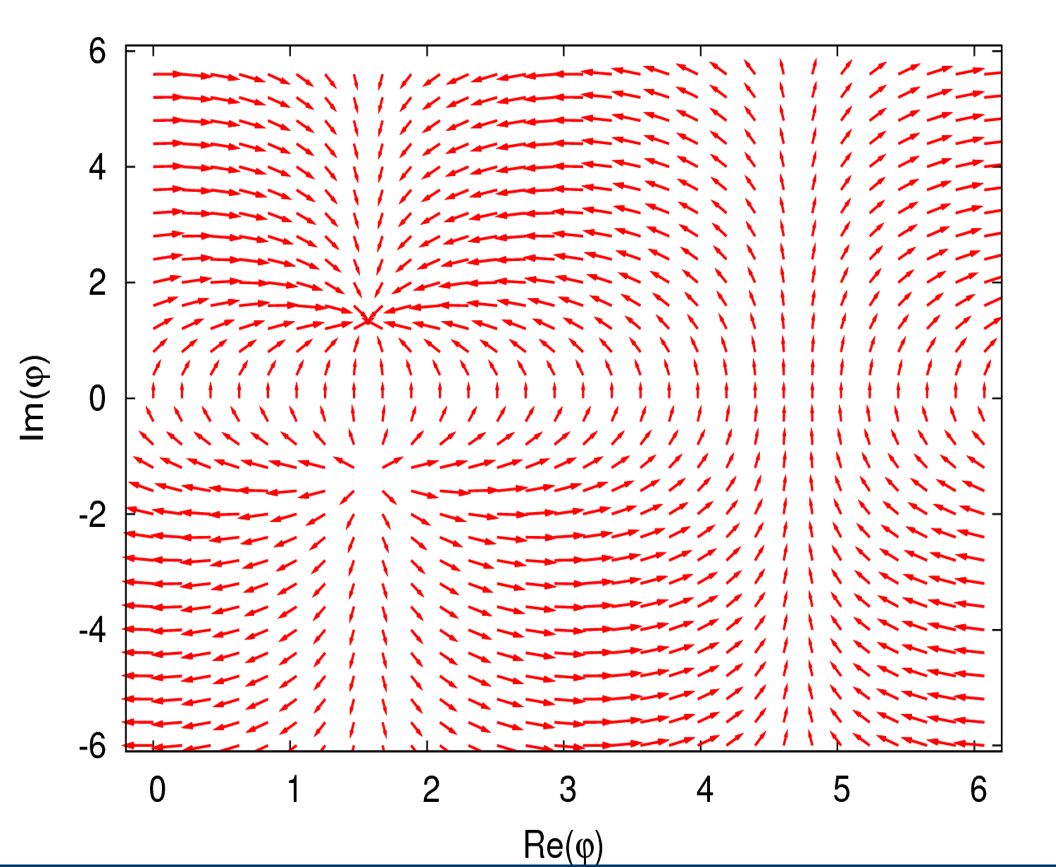
Yes: real (SU(2)) averages
Schwinger-Dyson equations
fulfilled

No general proof of convergence

But Schwhinger-Dyson eqs. are fulfilled

Runaway trajectories present

Typical drift structure



Noise is real "horizontal"

Runaway if field stays at $\frac{3}{2}\pi$

In continuum probabilty of a runaway=0

Discretised: getting far away

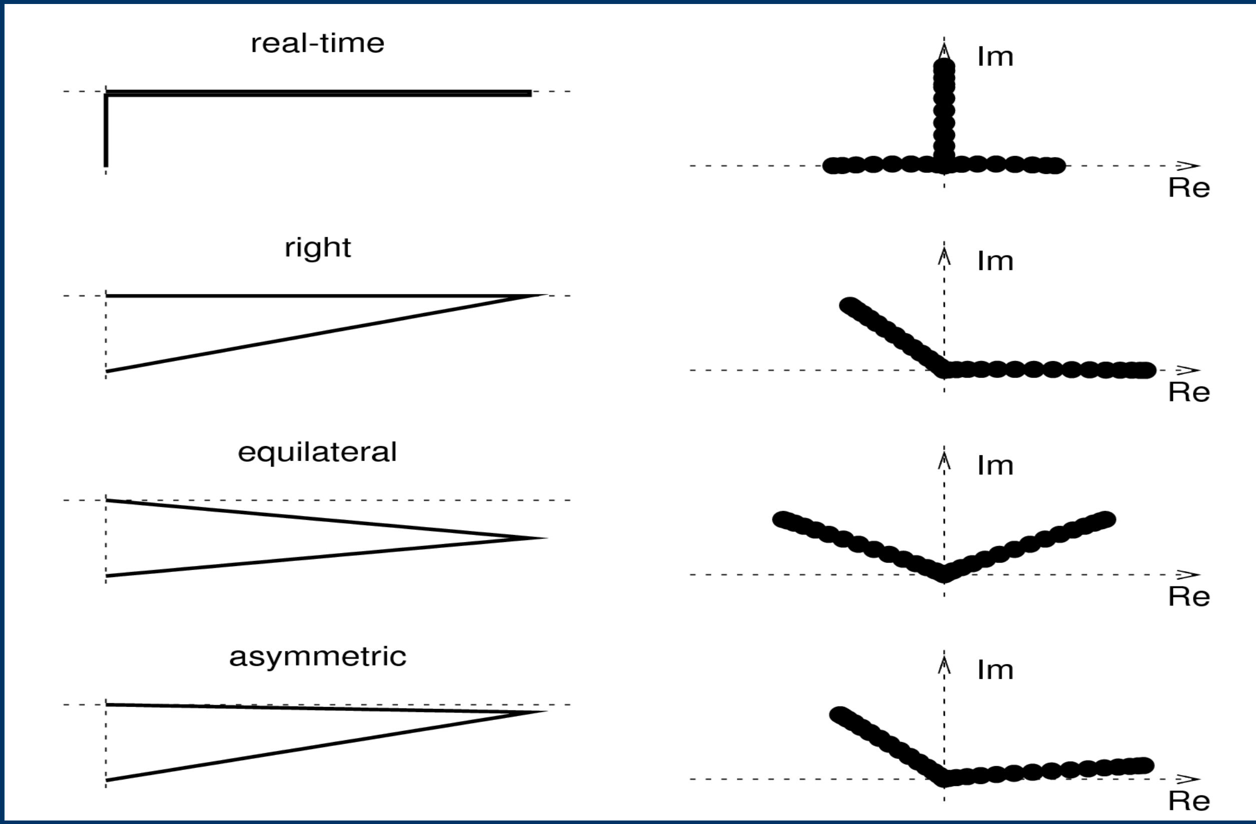
Numerical problem
drift proportional to field

Solution: small stepsize
restarting with different seed

Continuum limit in $\Delta\tau$ is reached

Action of the langevin method=
path integral on a complex time-contour

downwards sloped
countour: regulator



Studied models:

-- quantum oscillator (0+1)D scalar field theory
-- SU(2) pure gauge (3+1)D field theory with Wilson action

J. Berges, Sz. Borsányi, D. Sexty, I.-O. Stamatescu PRD75 045007

-- few variable toy models: U(1) one-plaquette models
SU(2) one-plaquette model

J. Berges, D. Sexty NPB799 306

Scalar Theory

Complex countour given by: C_t , $\Delta_t = C_{t+1} - C_t$, $C_0 = 0,\ C_{N_t} = -i\beta$

action discretised on the contour

$$S = \sum_t \left(\frac{(\phi_{t+1} - \phi_t)^2}{2\Delta_t} - \Delta_t \frac{V(\phi_t) + V(\phi_{t+1})}{2} \right)$$

Langevin updating in "5th" coordinate

$$\frac{d\phi_t}{d\tau} = i \frac{\partial S}{\partial \phi_t} + \eta_t(\tau)$$

$$\langle \eta_t(\tau) \rangle = 0$$

$$\langle \eta_t(\tau) \eta_{t'}(\tau') \rangle = 2\delta(\tau - \tau')\delta_{tt'}$$

discretised:

$$\phi_t(\tau + \epsilon) = \phi_t(\tau) + i\epsilon \frac{\partial S}{\partial \phi_t} + \sqrt{\epsilon}\, \eta_t(\tau)$$

Interacting scalar oscillator

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4$$

Thermal equilibrium $\longrightarrow$ periodic boundary conditions

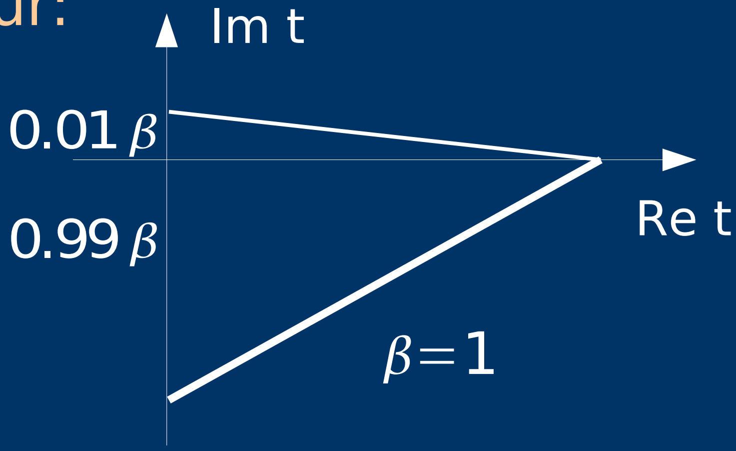
$$\phi_0 = \phi_{N_t}$$

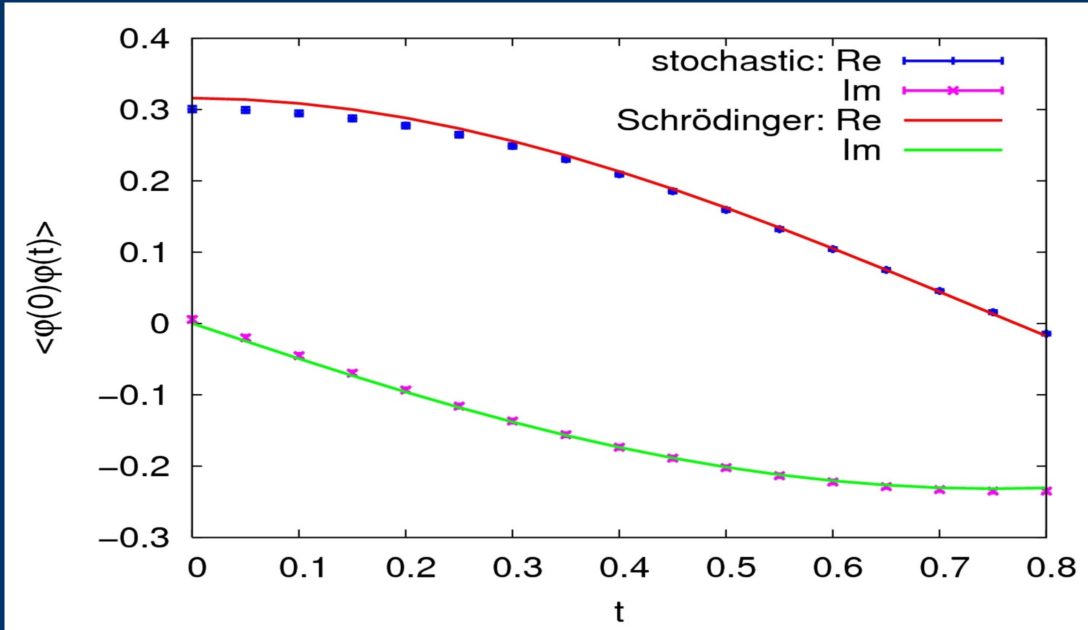
Real-time two point function of quantum oscillator

Thermal equilibrium:
periodic boundary cond.

Imaginary extent gives $\beta = \frac{1}{T}$

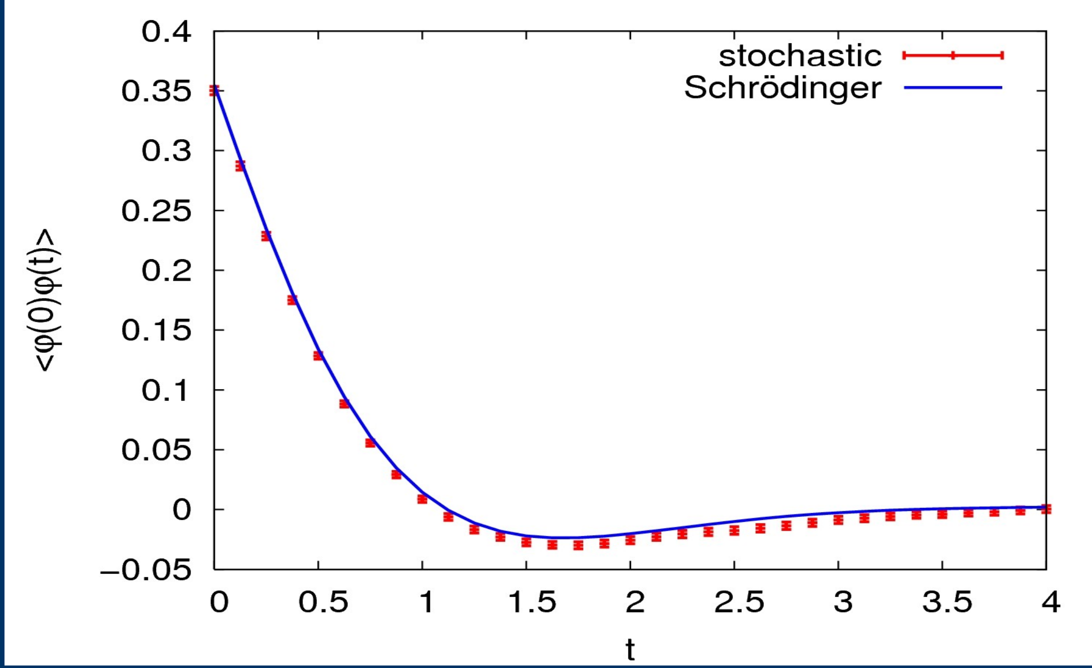
Asymmetric contour:





Smaller temperature
longer contour

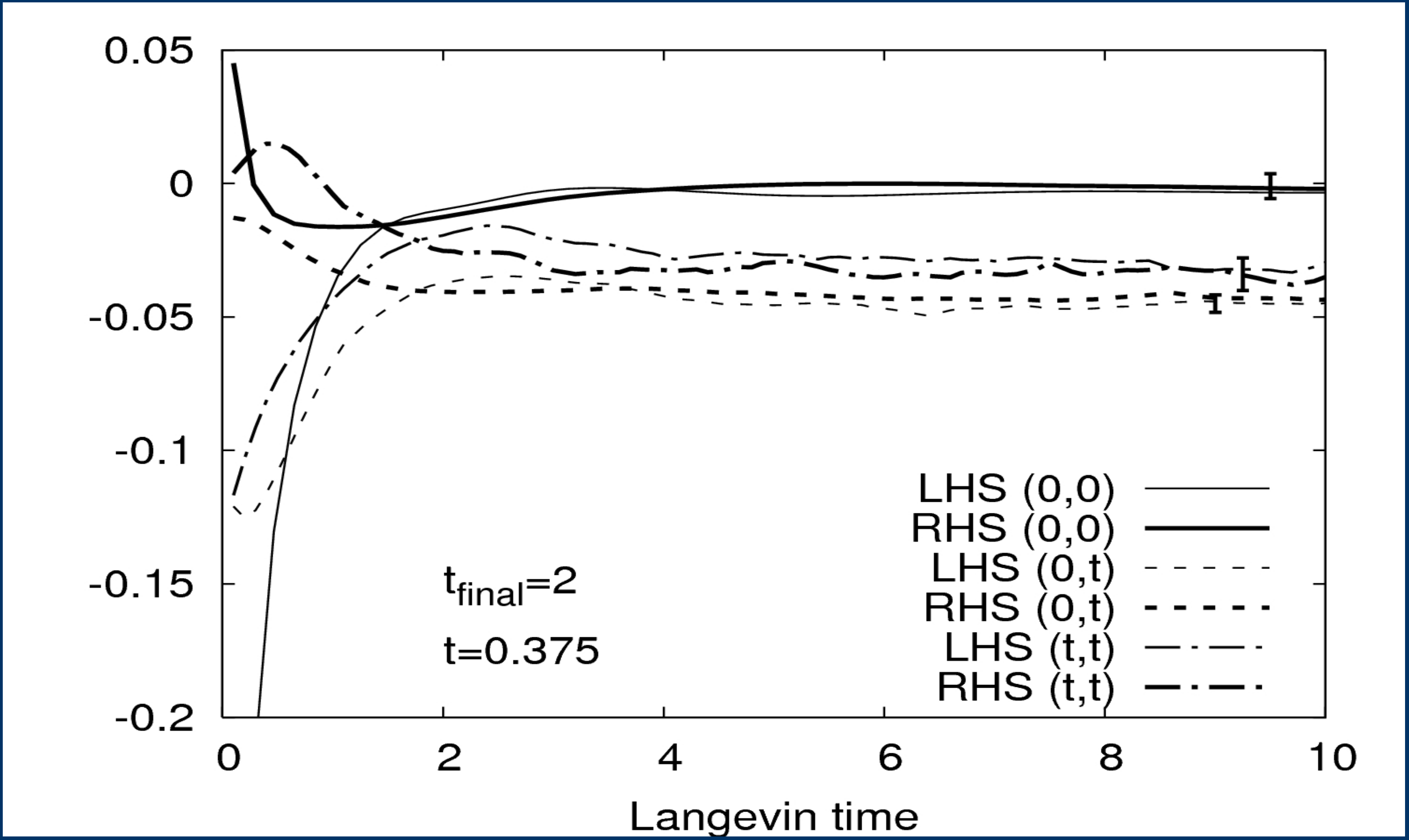
$\beta = 8$



Reproduces the Schrodinger equation result.

Numerical check of the Schwinger-Dyson equation

$$\sum_{\tau} G_{0,t\tau}^{-1} \langle \varphi_\tau \varphi_{t'} \rangle - \delta_{tt'} = -i\lambda \langle \varphi_t \varphi_t \varphi_t \varphi_{t'} \rangle$$



Non-equilibrium time evolution

Generating functional with initial density matrix:

$$Z(J,\rho)=Tr\left(\rho T_C e^{i\int_c J(x)\Phi(x)}\right)=\int d\varphi_1 d\varphi_2 \rho(\varphi_1,\varphi_2)\int_{\varphi_1}^{\varphi_2} D'\varphi\, e^{i\int_c L(x)+J(x)\varphi(x)}$$

Exponentializing the density matrix

Including φ_1,φ_2 in the path integral

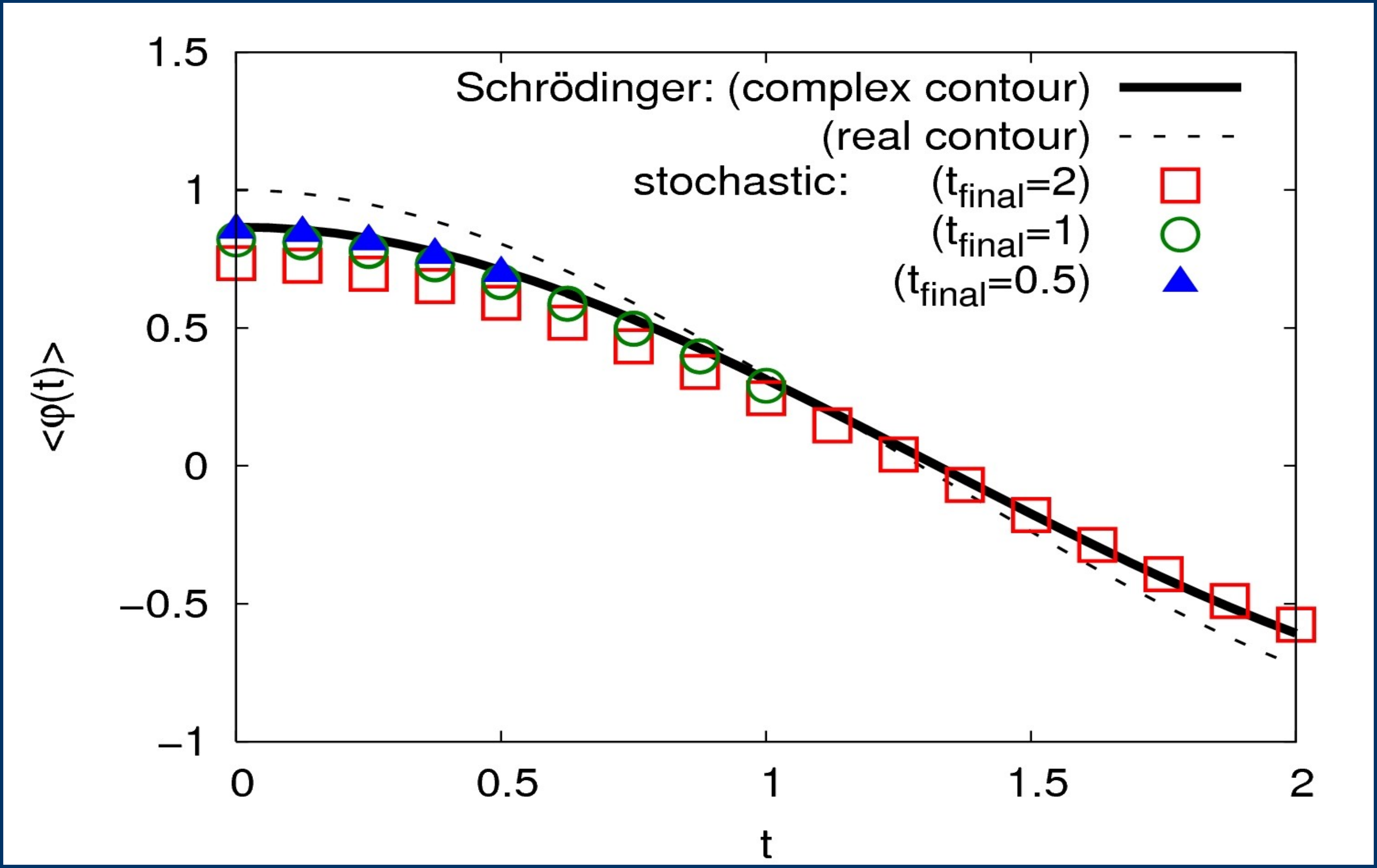
$$\langle A(\varphi)\rangle=\int D\varphi_u D\varphi_l \exp(iS_\rho(\varphi_u,\varphi_l))\,A(\varphi_u)$$

Langevin simulation with new “action”: $S_\rho[\varphi_u,\varphi_l]=S[\varphi_u]-S[\varphi_l]-\frac{i}{a_t}S_0(\varphi_u,\varphi_l)$

Most general gaussian density matrix with 5 parameters:

$$S_0(\varphi_u,\varphi_l)=i\dot{\phi}(\varphi_u-\varphi_l)-\frac{\sigma^2+1}{8\xi^2}\left((\varphi_u-\phi)^2+(\varphi_l-\phi)^2\right)$$
$$+\frac{i\eta}{2\xi}\left((\varphi_u-\phi)^2-(\varphi_l-\phi)^2\right)+\frac{\sigma^2-1}{4\xi^2}(\varphi_u-\phi)(\varphi_l-\phi)$$

Non-equilibrium time evolution



Contour with 5% slope

Bigger real time extent ⟶ worse agreement

SU(2) pure gauge theory

Wilson action:

$$S=-\beta_0\sum_{x,i}\frac{1}{2\,Tr\,\mathbf{1}}\left(Tr\,U_{x,0i}+Tr\,U^{-1}_{x,0i}\right)-1$$
$$+\beta_s\sum_{x,i<j}\frac{1}{2\,Tr\,\mathbf{1}}\left(Tr\,U_{x,ij}+Tr\,U^{-1}_{x,ij}\right)-1$$

$$\beta_0=\frac{2\,Tr\,\mathbf{1}\,a_s}{g_0^2 a_t}$$

$$\beta_s=\frac{2\,Tr\,\mathbf{1}\,a_t}{g_0^2 a_s}$$

Updating the link variables:

$$U'_{x,\mu}=\exp\left(i\lambda_a\left(\epsilon\, i\, D_{x\mu a}S[U]+\sqrt{\epsilon}\,\eta_{x\mu a}\right)\right)U_{x\mu}$$

$$\langle\eta_{x\mu a}\rangle=0$$
$$\langle\eta_{x\mu a}\eta_{y\nu b}\rangle=2\,\delta_{xy}\,\delta_{\mu\nu}\,\delta_{ab}$$

Left derivative: $$D_a f(U)=\left(\frac{\partial}{\partial\alpha}f\left(e^{i\lambda_a\alpha}U\right)\right)\Big|_{\alpha=0}$$

$$U=\exp\left(i\frac{\varphi\,\hat{n}\hat{\sigma}}{2}\right)=\left(\cos\frac{\varphi}{2}\right)\mathbf{1}+i\left(\sin\frac{\varphi}{2}\right)\hat{n}\hat{\sigma}$$
$$U=a\mathbf{1}+i\,b_i\sigma_i \qquad a^2+b_i b_i=1$$

complexifed link variables

SU(2) $\longrightarrow$ SL(2,C)

compact $\longrightarrow$ non-compact

$a,\ b_i$ become complex variables

Schwinger Dyson equations for lattice gauge theory

Langevin-time equilibrium reached:

$$\langle U_{x\mu a}(\tau+d\tau)\rangle = \langle U_{x\mu a}(\tau)\rangle \quad \Rightarrow \quad \langle D_{x\mu a} S\rangle = 0$$

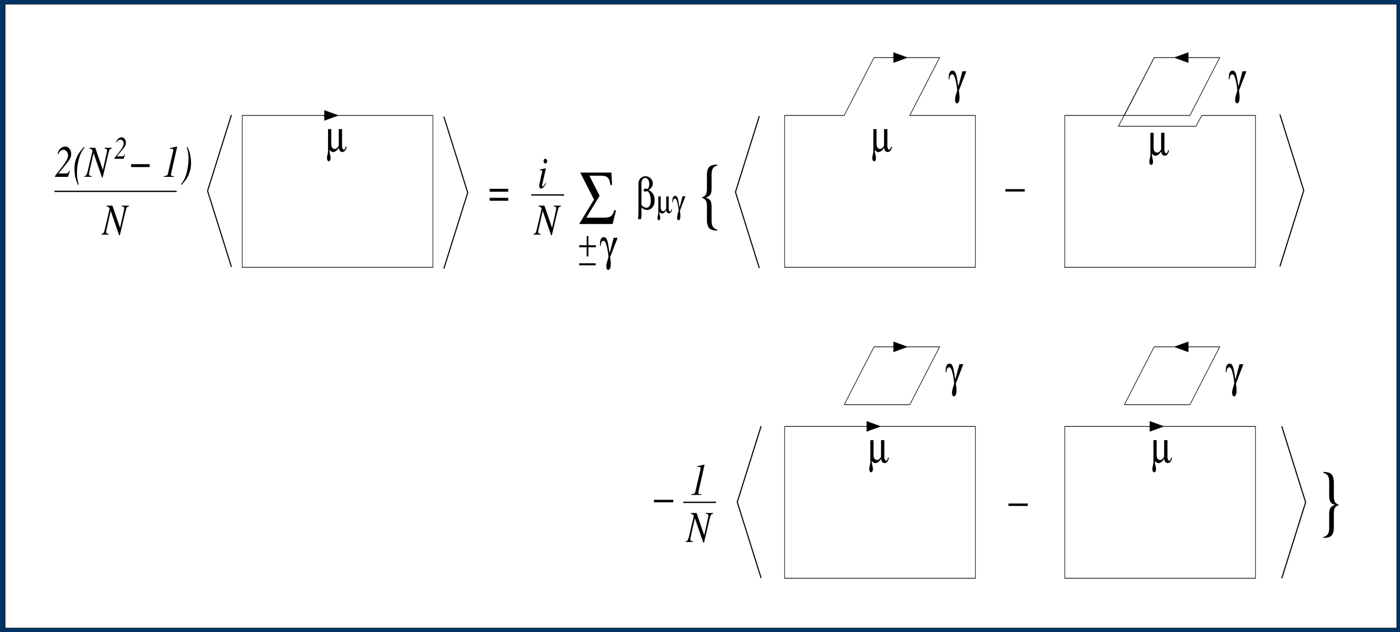
First Schwinger Dyson equation

Plaquette average is Langevin time independent

$$\langle U_{x,\mu\nu}(\tau+d\tau)\rangle = \langle U_{x,\mu\nu}(\tau)\rangle$$

Schwinger Dyson equation for plaquette average

can also be derived using the properties of Haar integration in the original integration over group space

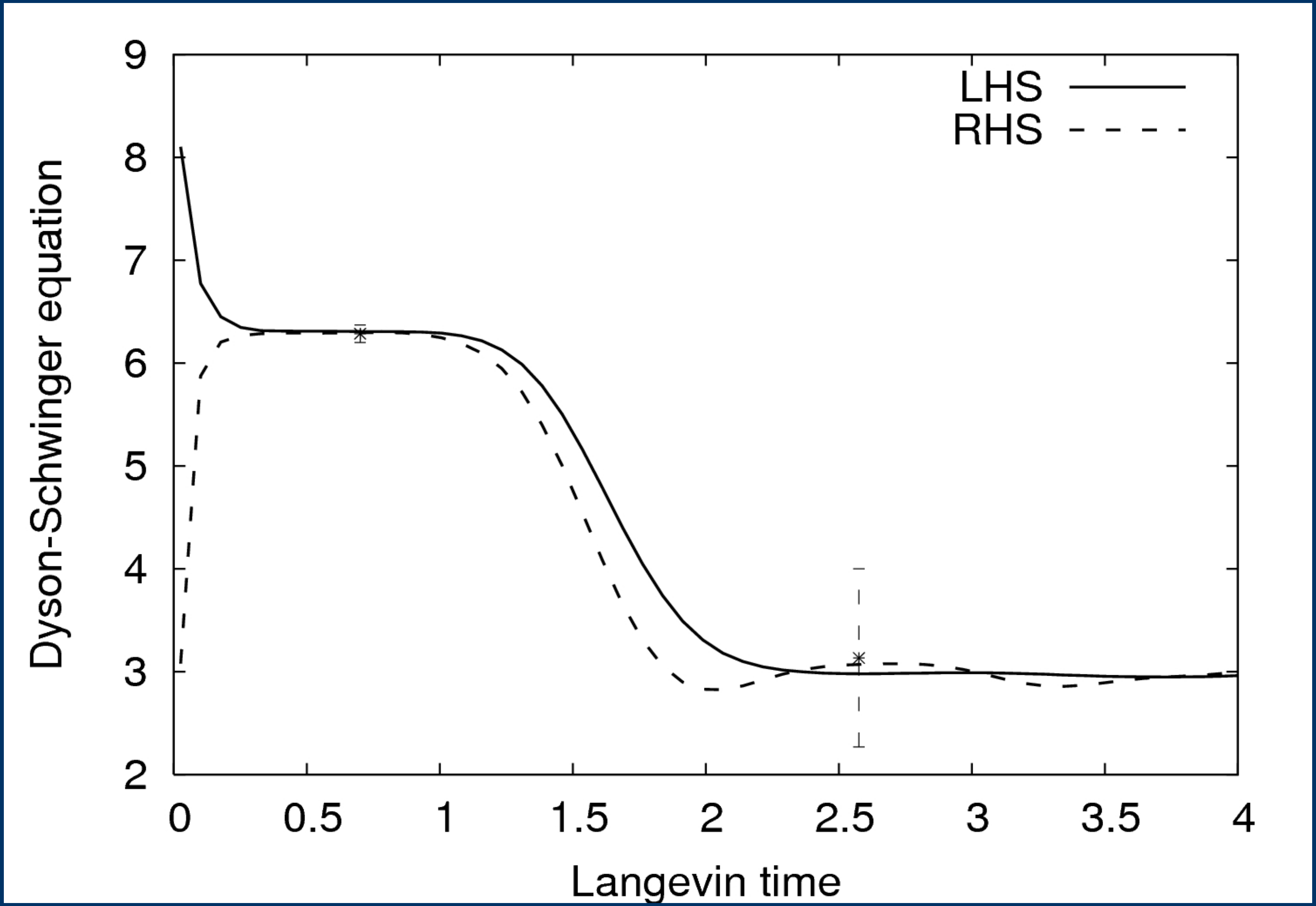


This method gives solutions of SD equations (all of them!)

(loophole: one might get unphysical solution)

SU(2) field theory

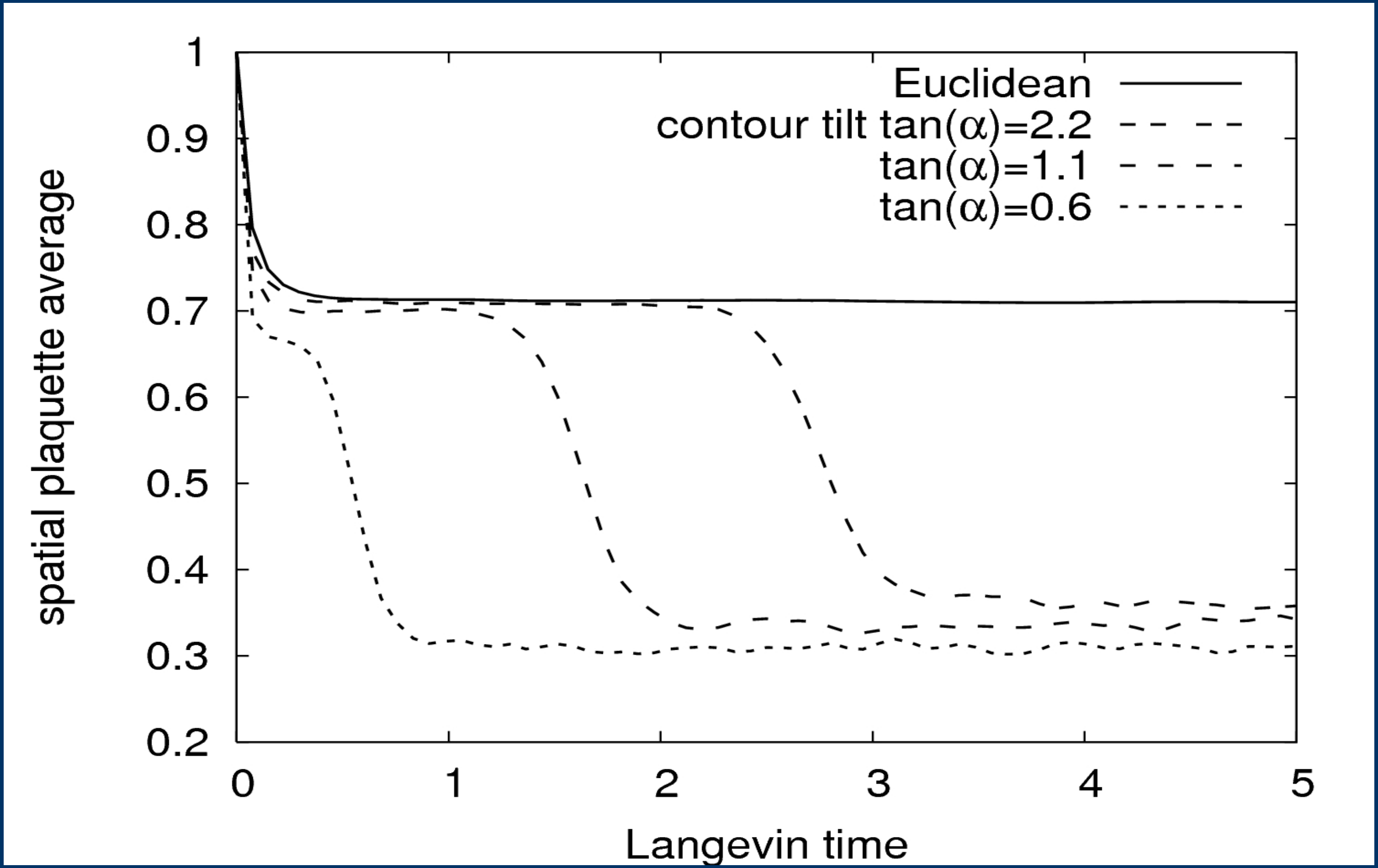
Numerical check of the Schwinger-Dyson equation



SD equations are fulfilled in both regions

SU(2) gauge theory without gaugefixing

without gauge fixing, non-physical fixpoint is always present



How to stabilize the first (physical) result?

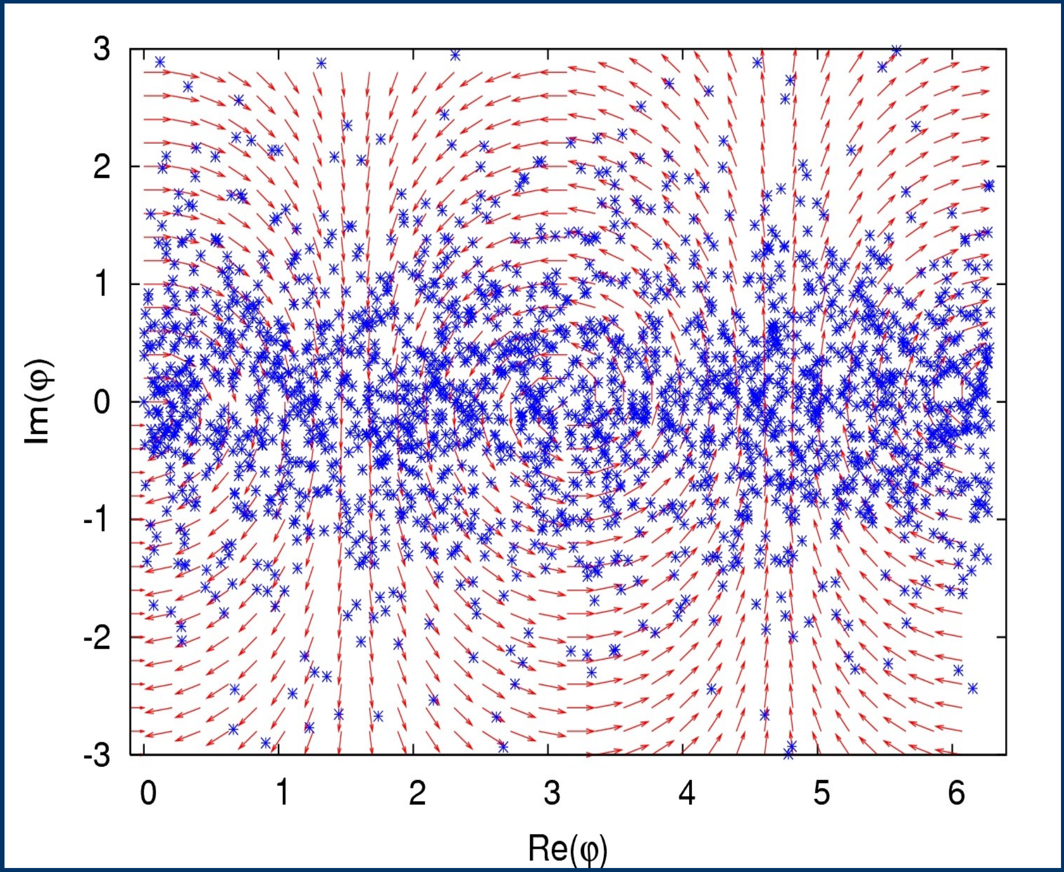
U(1) One plaquette model

$S_0 = i\beta\cos(\varphi)$

We are interested in averages: $\langle f(\varphi)\rangle = \frac{1}{Z}\int_0^{2\pi} d\varphi\, e^{i\beta\cos\varphi} f(\varphi)$

Langevin equation: $\frac{d\varphi}{d\tau} = -i\beta\sin\varphi + \eta(\tau)$

Distribution of φ on the complex plane



Failure of the naïve method

exact result: $\langle e^{i\varphi}\rangle = i\,0.575$

stochastic result:
$-0.009 \pm 0.006 + i(0.00006 \pm 0.00007)$

symmetric distribution
result compatible with zero

Stochastic reweighting

generalization: $S_p = i\beta\cos(\varphi) + i p\varphi$

$$\langle O \rangle_p = \frac{1}{Z_p}\int_0^{2\pi} d\varphi\, e^{S_p} O(\varphi)$$

Langevin equation: $\frac{d\varphi}{d\tau} = -i\beta\sin\varphi + i p + \eta(\tau)$

reweighting factor: $\omega_p = \exp(S_0 - S_p)$

Reweigting formula

averages with S_0 calculated
from averages with S_p

$$\langle O \rangle_0 = \frac{\int_0^{2\pi} d\varphi\, e^{iS_p}\omega_p O(p)}{\int_0^{2\pi} d\varphi\, e^{iS_p}\omega_p} = \frac{\langle \omega_p O \rangle_p}{\langle \omega_p \rangle_p}$$

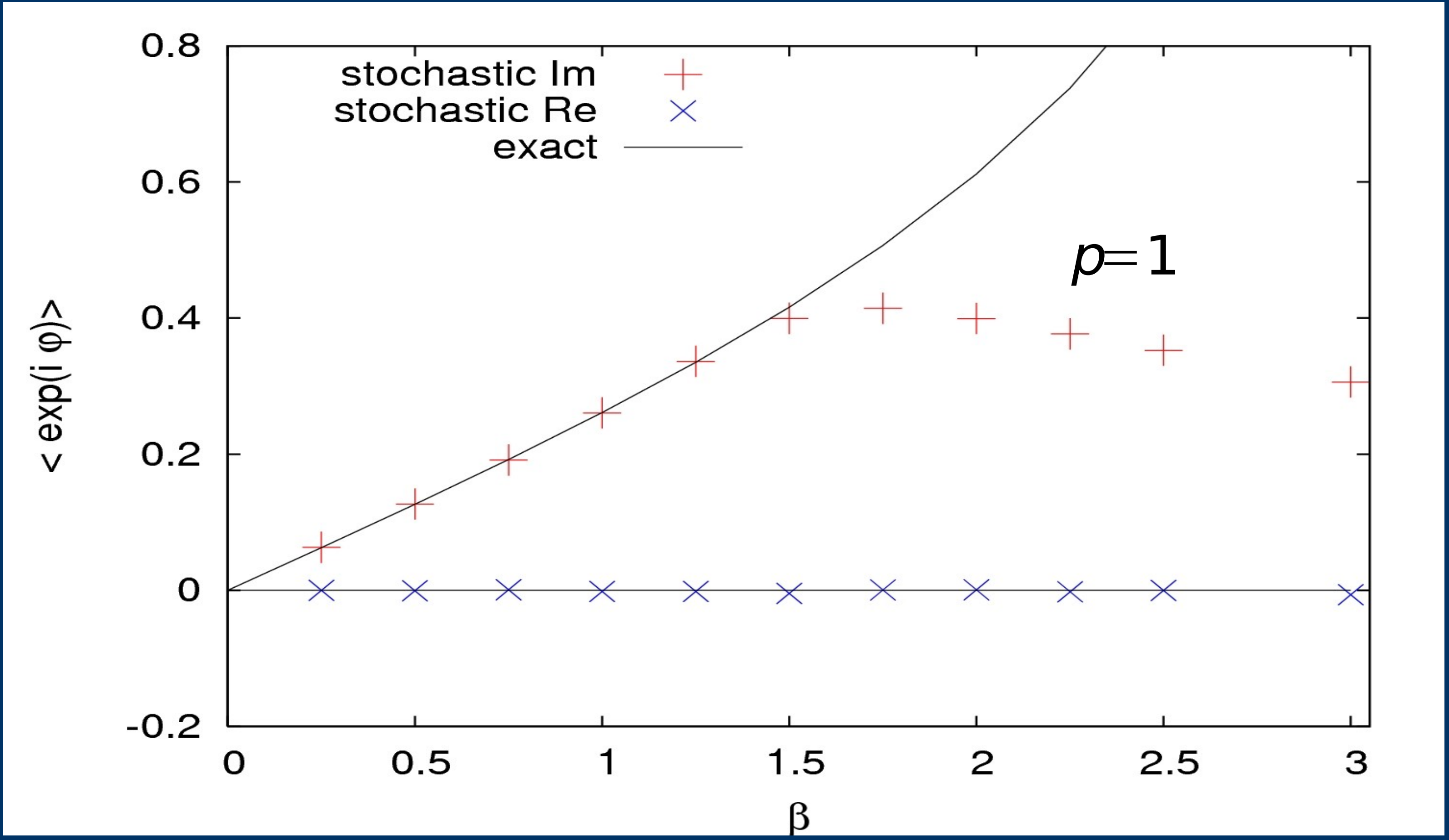
$$\langle e^{i\varphi} \rangle_0 = \frac{\langle 1 \rangle_{p=1}}{\langle e^{-i\varphi} \rangle_{p=1}} = (-0.02 \pm 0.02) + i(0.574 \pm 0.001)$$

Exact result: $\langle e^{i\varphi} \rangle_{p=0} = i\,0.575$

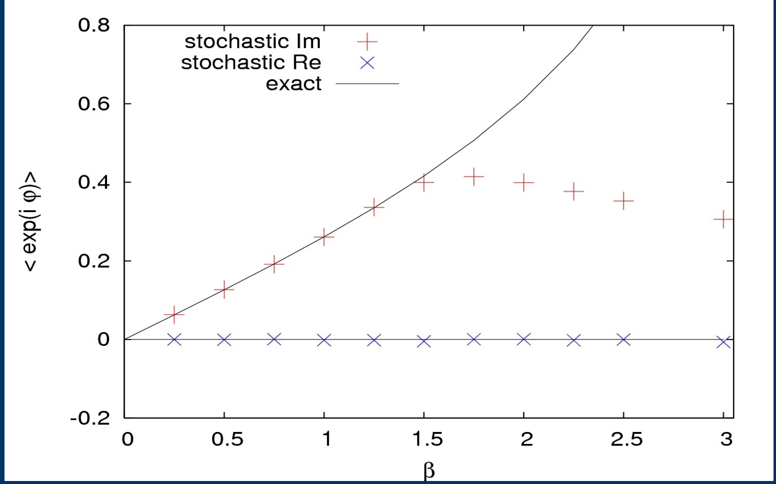
with reweighting it works!

Using the generalized action $S_p = i\beta\cos(\varphi) + ip\varphi$

Correct results obtained for $\langle \exp(i\varphi) \rangle$ in the region: $\beta \leq p$



Flowchart: normalized drift vectors on the complex plane



shows fixedpoint (zero drift term) structure on the complex φ plane

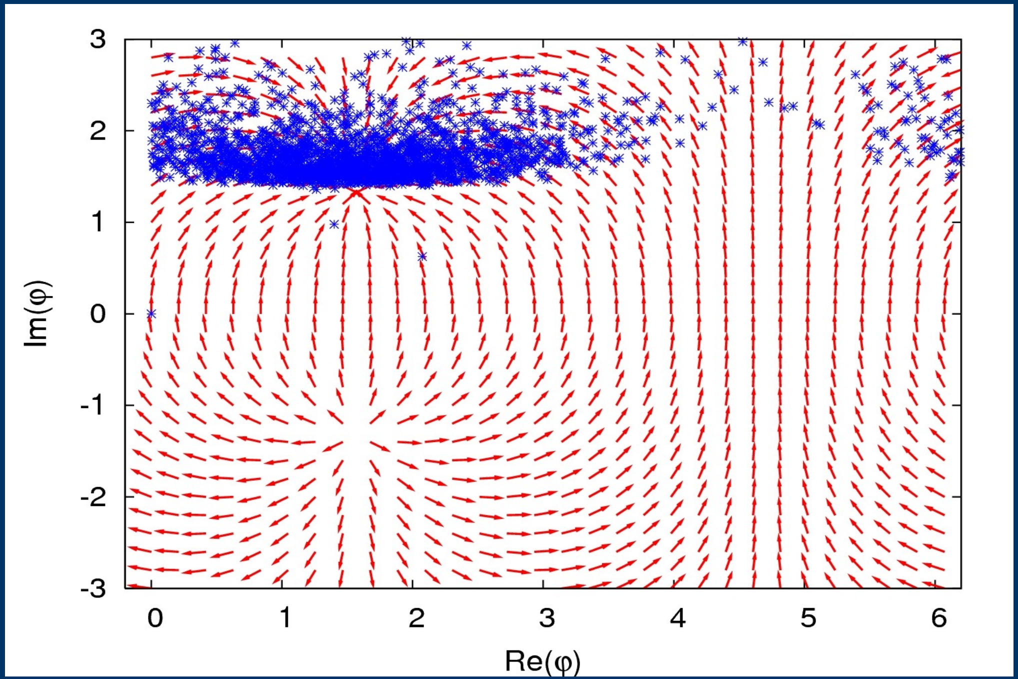
Attractive fixedpoint present

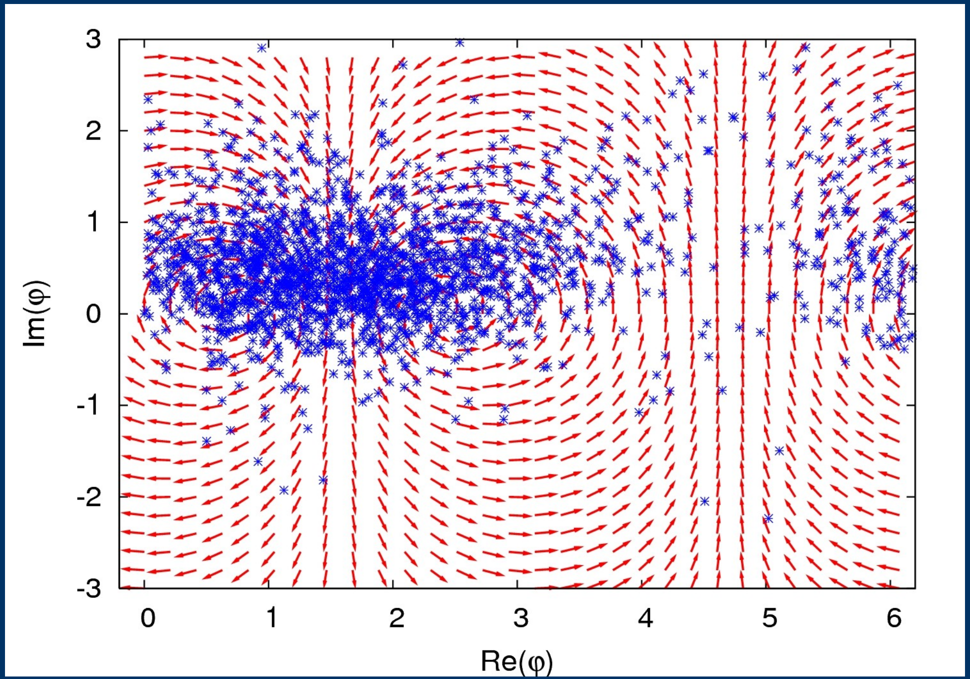
smaller distribution
correct results

$\beta=0.5, p=1$

No attractive fixedpoint present
(only indifferent)
larger distribution
incorrect results

$\beta=1.5, p=1$





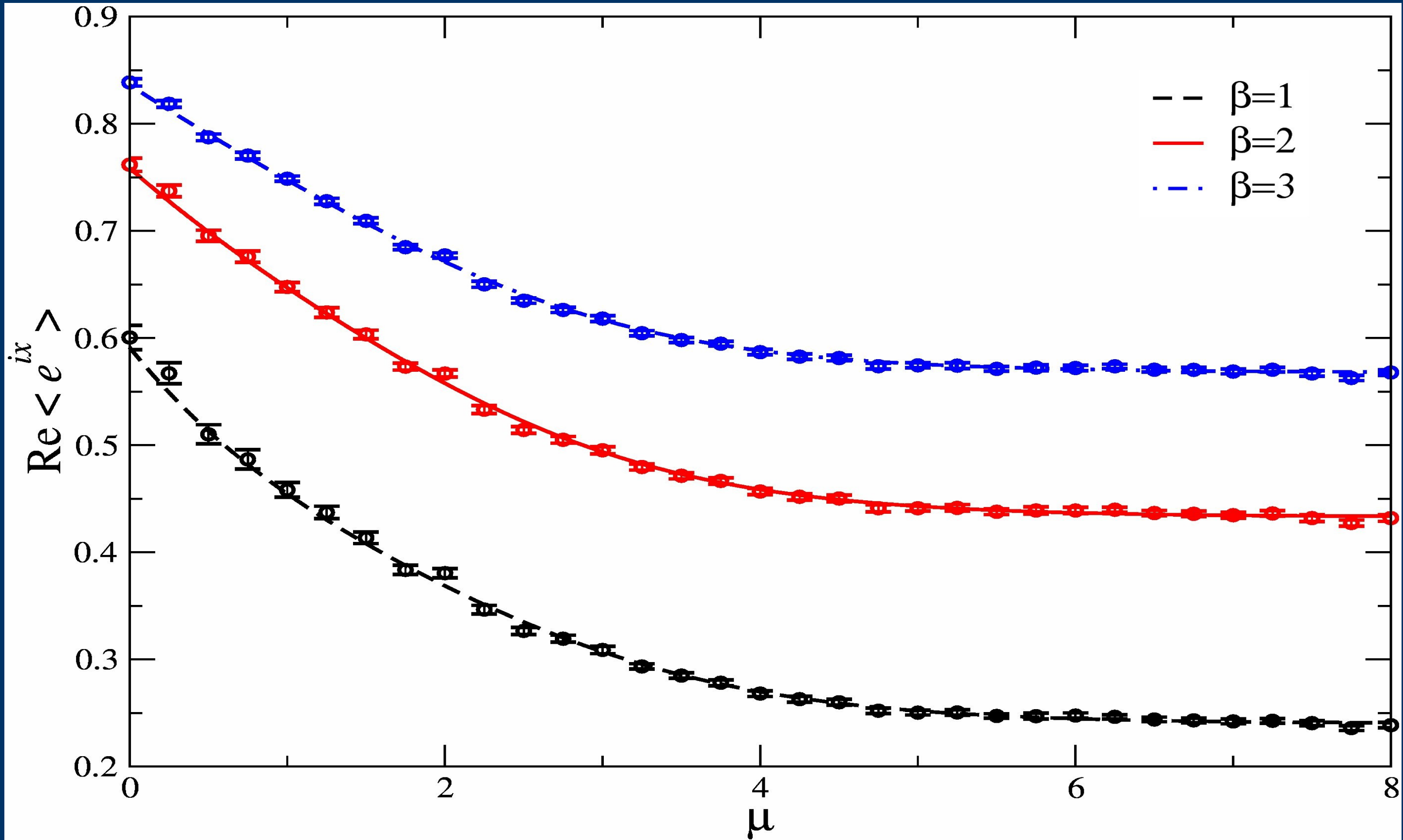
Euclidean U(1) One plaquette model

with "fermion determinant" Aarts, Stamatescu 2008

Simple model of QCD with finite chemical potential

$$Z=\int_0^{2\pi} dx\, e^{-S_B} \det M \qquad S_B=-\frac{\beta}{2}(U+U^{-1})=-\beta\cos(x) \qquad U=e^{ix}$$

$$\det M = 1+\frac{1}{2}\kappa(e^{\mu}U+e^{-\mu}U^{-1})=1+\kappa\cos(x-i\mu)$$



Fixedpoint structure

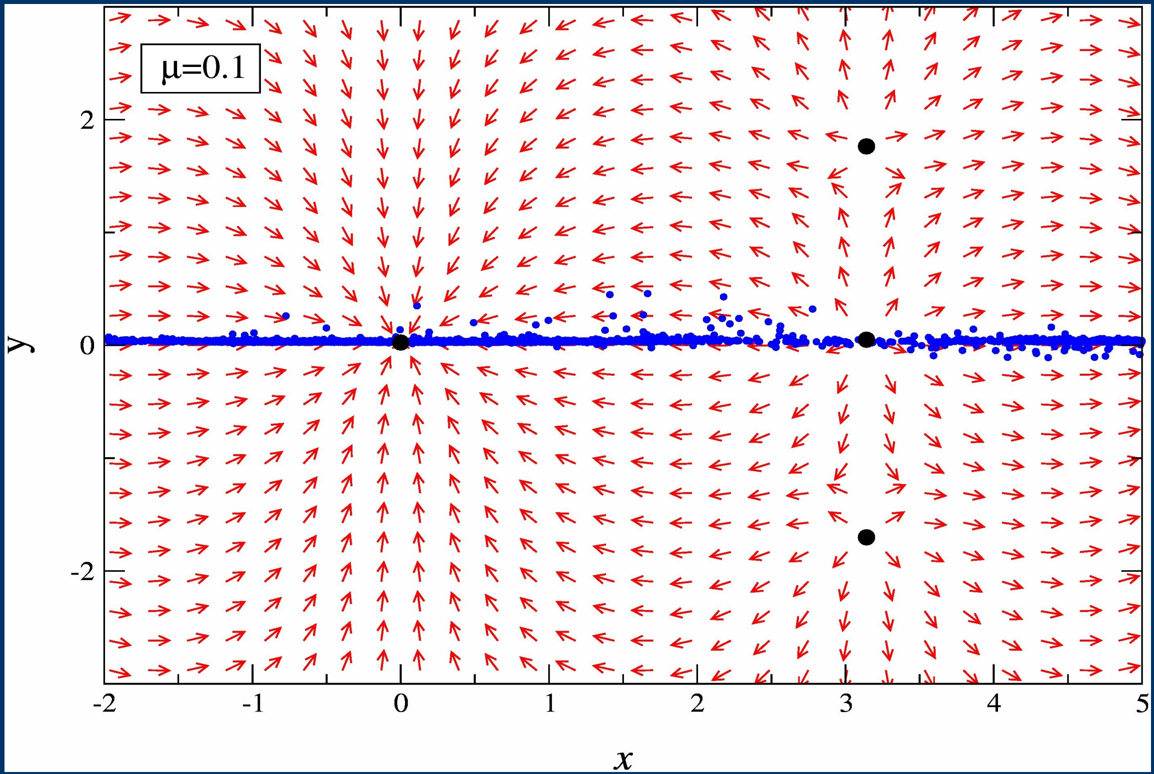
Distribution centered
around attractive fixedpoints
of the flow

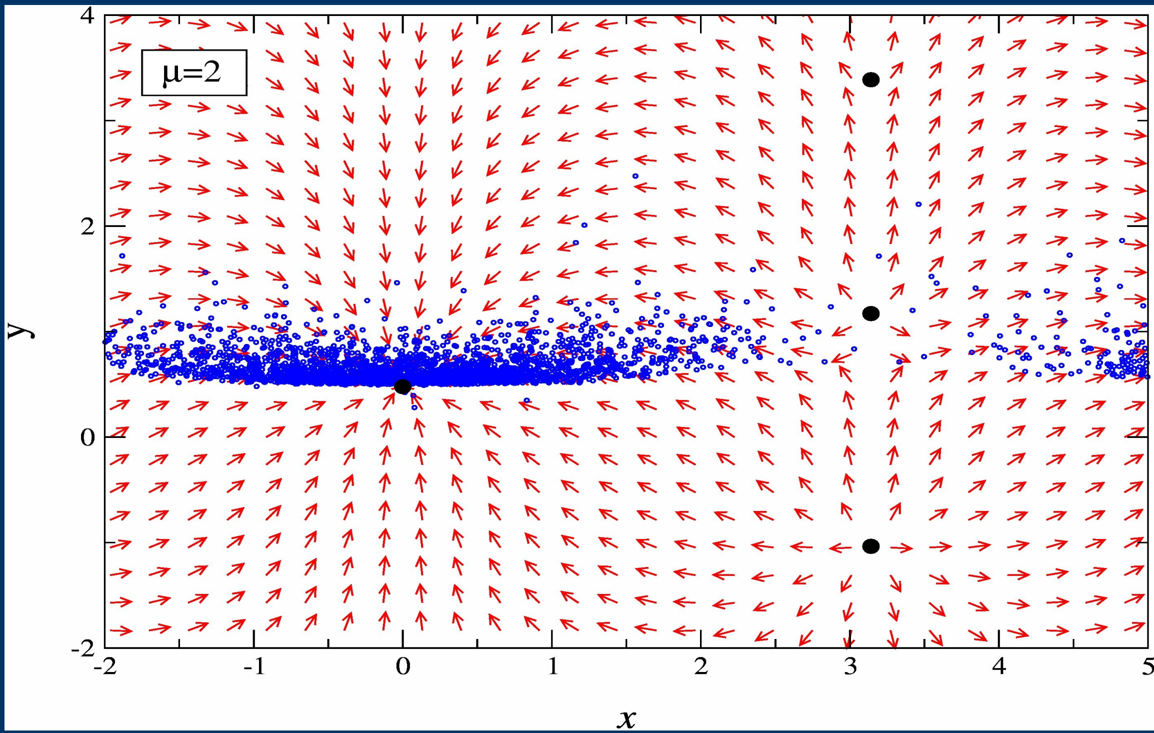
μ grows

↓

Fixed points move
no change in analytical structure

No breakdown,
Langevin works for high μ



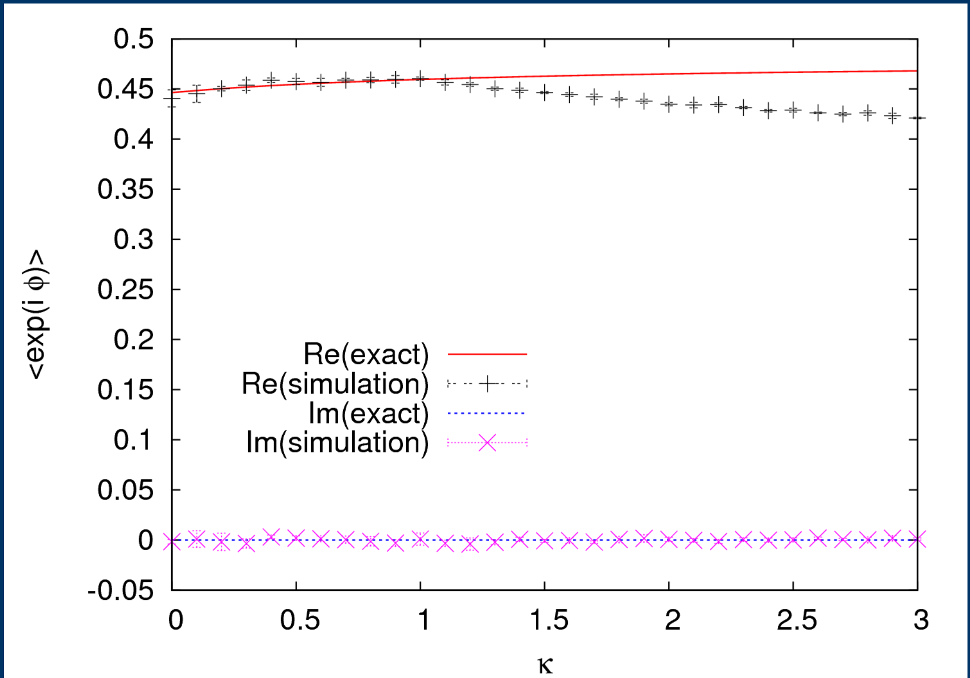


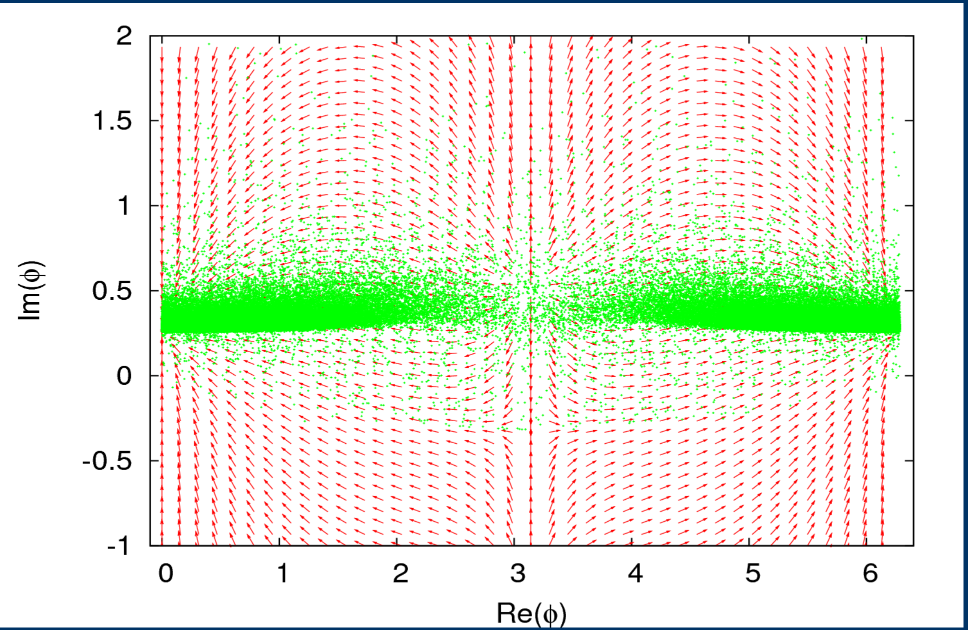
Breakdown at high kappa

Berges, Sexty in preparation

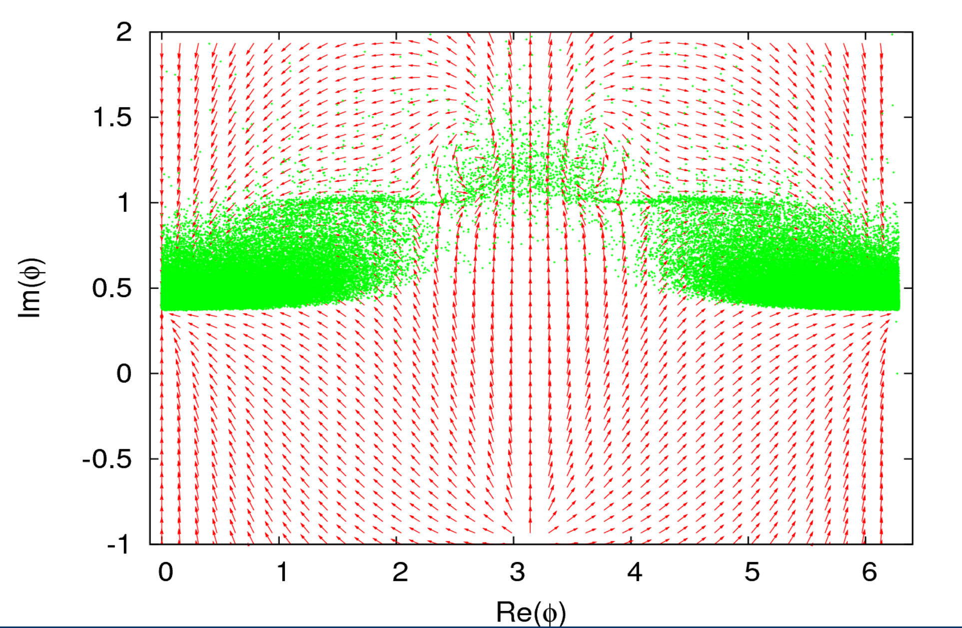
At $\kappa = 1$ the fixedpoint structure changes

A new attractive fixedpoint sucks away part of the distribution





$\kappa = 0.5$



$\kappa = 1.5$

Gaugefixing in SU(2) one plaquette model

SU(2) one plaquette model: $S = i\beta \, Tr \, U \qquad U \in SU(2)$

"gauge" symmetry: $U \rightarrow W U W^{-1}$ complexified theory: $U, W \in SL(2, \mathbb{C})$

exact averages by numerical integration: $\langle f(U) \rangle = \frac{1}{Z} \int_0^{2\pi} d\varphi \int d\Omega \sin^2 \frac{\varphi}{2} e^{i\beta \cos\frac{\varphi}{2}} f(U(\varphi, \hat{n}))$

Langevin updating $U' = \exp\left(i \lambda_a (\epsilon i D_a S[U] + \sqrt{\epsilon} \eta_a) \right) U$

parametrized with Pauli matrices

$$U = \exp\left(i \frac{\varphi \hat{n} \hat{\sigma}}{2} \right) = \left(\cos\frac{\varphi}{2} \right) \mathbf{1} + i \left(\sin\frac{\varphi}{2} \right) \hat{n} \hat{\sigma}$$

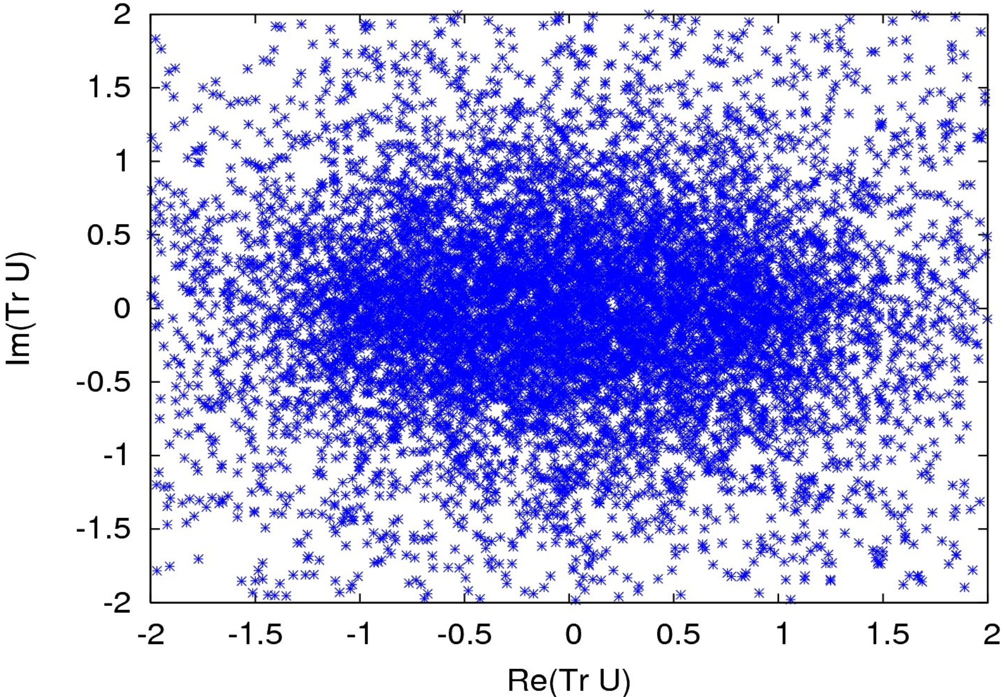
$$U = a \mathbf{1} + i b_i \sigma_i \qquad a^2 + b_i b_i = 1$$

After each Langevin timestep: fix gauge condition

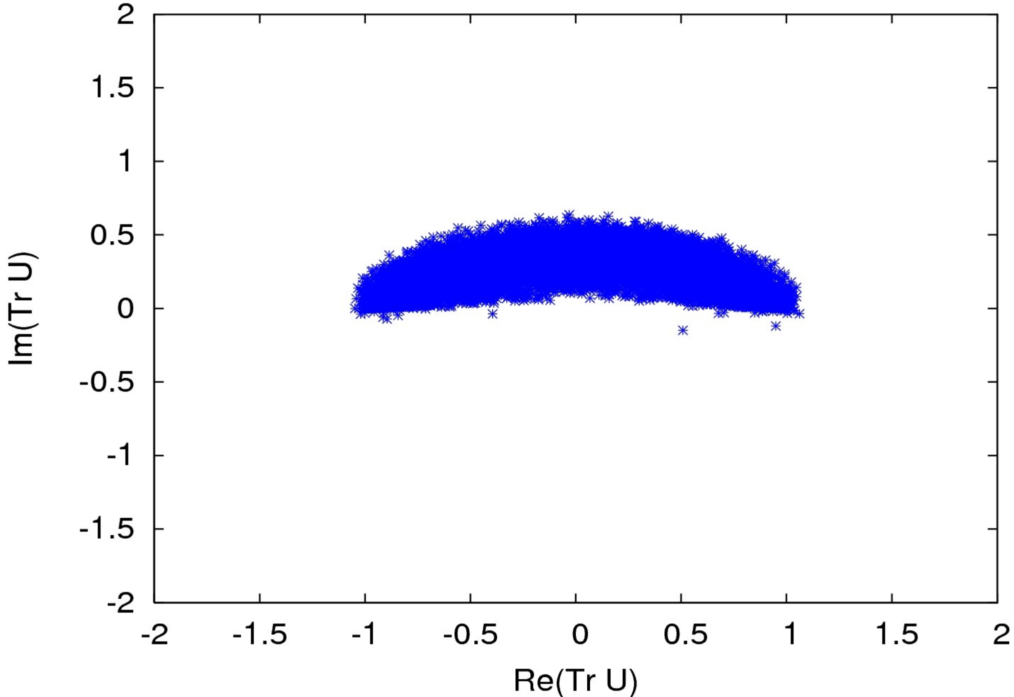
$$U = a \mathbf{1} + i \sqrt{1 - a^2} \sigma_3 \qquad b_i = (0, 0, \sqrt{1 - a^2})$$

SU(2) one-plaquette model

Distributions of Tr(U) on the complex plane



Without gaugefixing



With gaugefixing

Exact result from integration: $\langle Tr\, U \rangle = i\, 0.2611$

From simulation:

$(-0.02 \pm 0.02) + i(-0.01 \pm 0.02)$

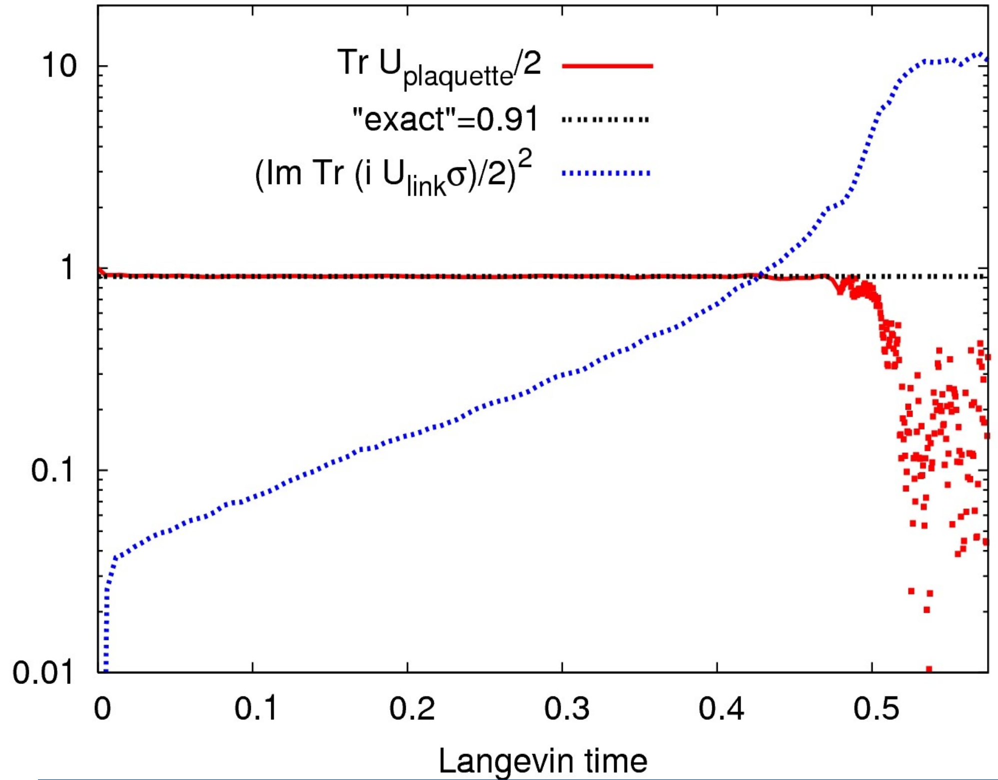
$(-0.004 \pm 0.006) + i(0.260 \pm 0.001)$

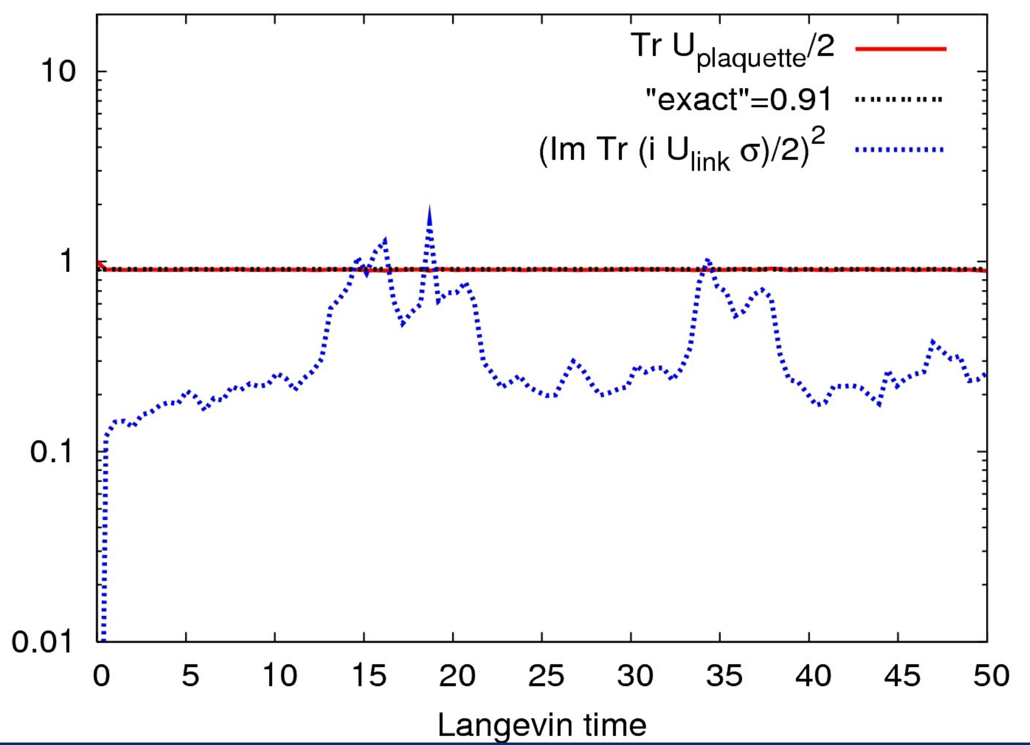
With gauge fixing, all averages are correctly reproduced

SU(2) field theory

$(Im Tr U)^2$ measures size of distribution

Without gauge fixing
non physical fixed point





Gauge fixing
small lattice coupling → large β

Correct result stabilizes

However:

Lattice coupling $g=0.5$

Scaling region $g \geq 1$

Conclusions

Real-time lattice results

Without optimization: short real time simulation of scalar oscillator in equilibrium and non-eq. gives correct results (Schrodinger)

Langevin method: Schwinger Dyson equation solver

Optimization methods to reduce fluctuations:
- reweighting
- gaugefixing
- using small lattice-coupling

Method gives physical solution for SU(2) lattice gauge theory

What is a complex contour with big slope useful for?

Spectral function reconstruction

[Nakahara, Asakawa, Hatsuda 99]
[Aarts, Resco 05]
[Jakovác, Petreczky, Petrov, Velytsky 06]

$$G(\tau)=\int \frac{d\omega}{2\pi} K(\tau,\omega)\rho(\omega)$$

$$K(\tau,\omega)=\frac{e^{\beta\omega}e^{-\tau\omega}+e^{\tau\omega}}{e^{\beta\omega}-1}=\frac{ch(\beta\frac{\omega}{2}-\tau\omega)}{sh\frac{\beta\omega}{2}}$$

Generalized for any $t\in\mathbb{C}$

$$K(t,\omega)=\frac{e^{\beta\omega}e^{-it\omega}+e^{it\omega}}{e^{\beta\omega}-1}$$

For $t\in\mathbb{R}$ → Fourier transformation, easily inverted

$$G(t)-G(-t)=\int_0^\infty \frac{d\omega}{2\pi}(e^{-it\omega}-e^{it\omega})\rho(\omega)$$

For $t=-i\tau$ → Usually underdetermined equations, iversion is hard
Some results with Maximum Entrophy Method.

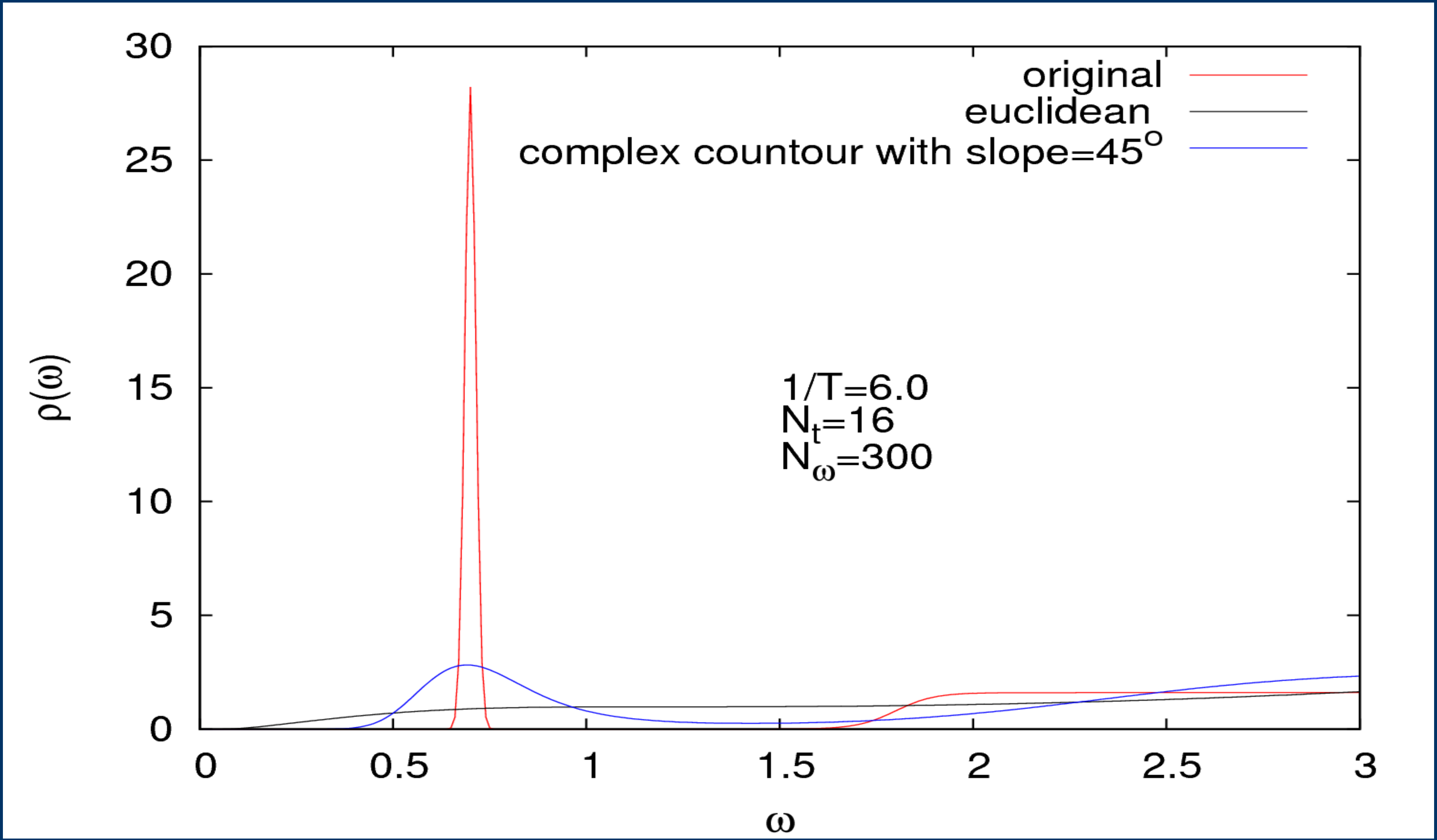
Spectral function reconstruction of mock data

Sloped contour $t \in \mathbb{C}$

Kernel Interpolates between real and euclidean

MEM is still used

Complex contour finds peak

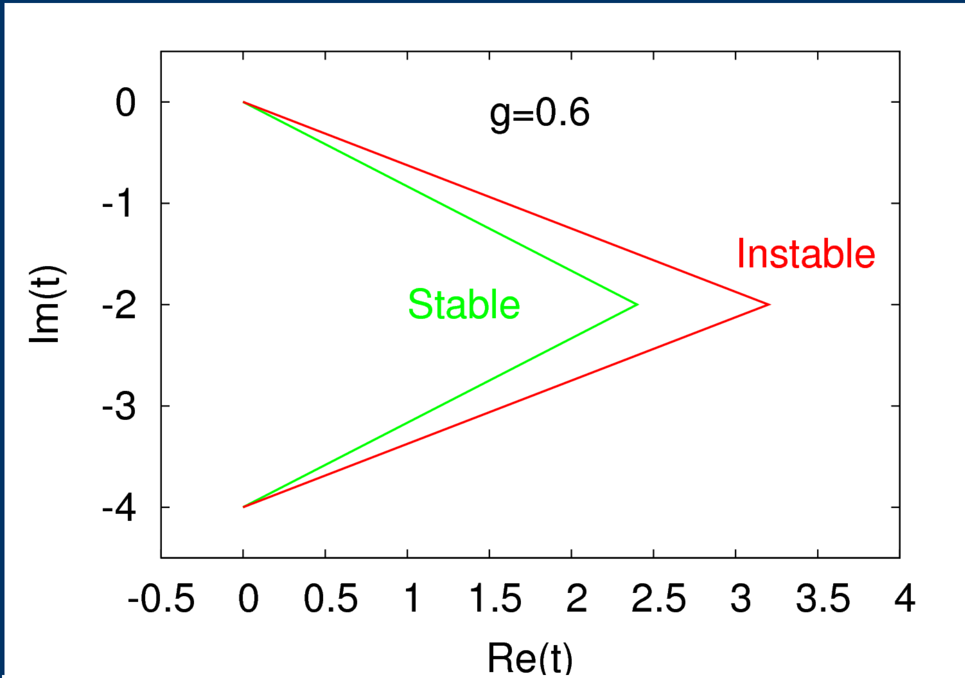


Accessible complex contours

$g=0.6$

Outside scaling region for practical lattice sizes

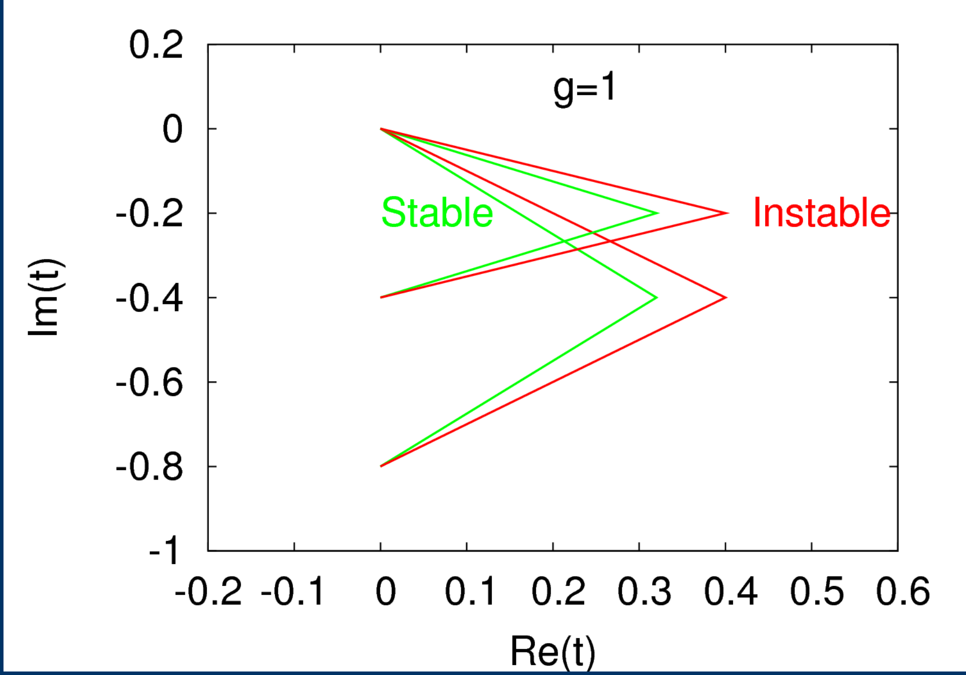
Low temperature



$g=1$

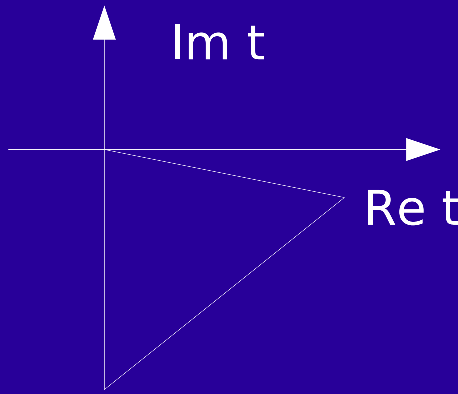
High temperature contours

Smaller slope is possible



Asymmetric contour

Lower temperature
Smaller slope



Scalar Theory

Complex countour given by: $C_t \;,\quad \Delta_t = C_{t+1} - C_t \;,\quad C_0 = 0,\; C_{N_t} = -i\beta$

action discretised on the contour $$S = \sum_t \left(\frac{(\phi_{t+1} - \phi_t)^2}{2\Delta_t} - \Delta_t \frac{V(\phi_t) + V(\phi_{t+1})}{2} \right)$$

Langevin updating in “5th” coordinate $$\frac{d\phi_t}{d\tau} = \frac{\partial S}{\partial \phi_t} + \eta_t(\tau)$$

Free theory: $V(\phi) = \frac{m^2 \phi^2}{2}$

The action can be diagonalized: $S = \frac{1}{2} \sum_a c^a \chi^a \chi^a, \qquad \chi^a = \sum_t \psi_t^a \phi_t$

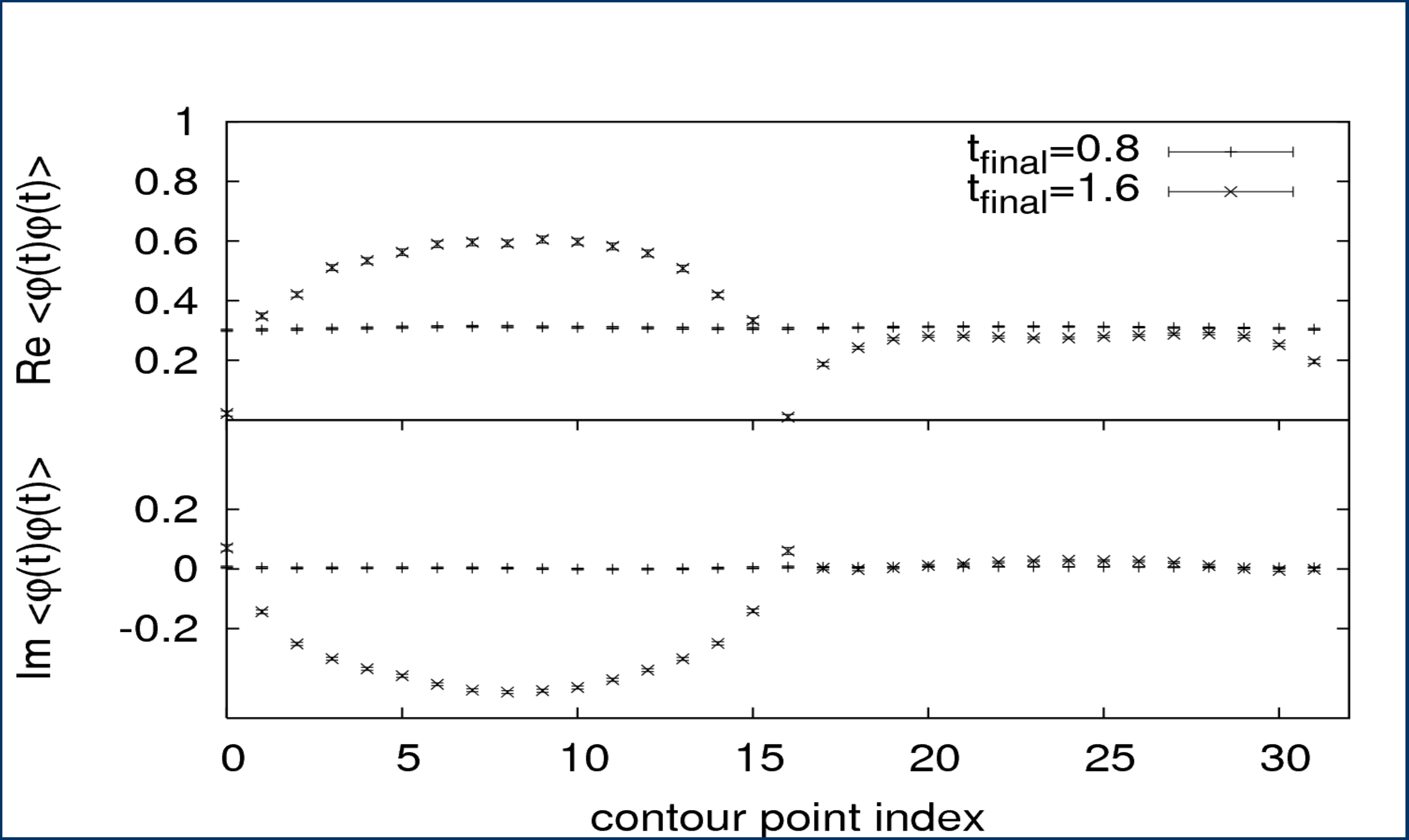
Langevin equation diagonalized coords.:

$$\frac{d\chi^a}{d\tau} = i c^a \chi^a + \eta^a$$

convergent if $Im(c^a) > 0$

Non-Physical fixpoints

long countour ► non-time translation invariant solution

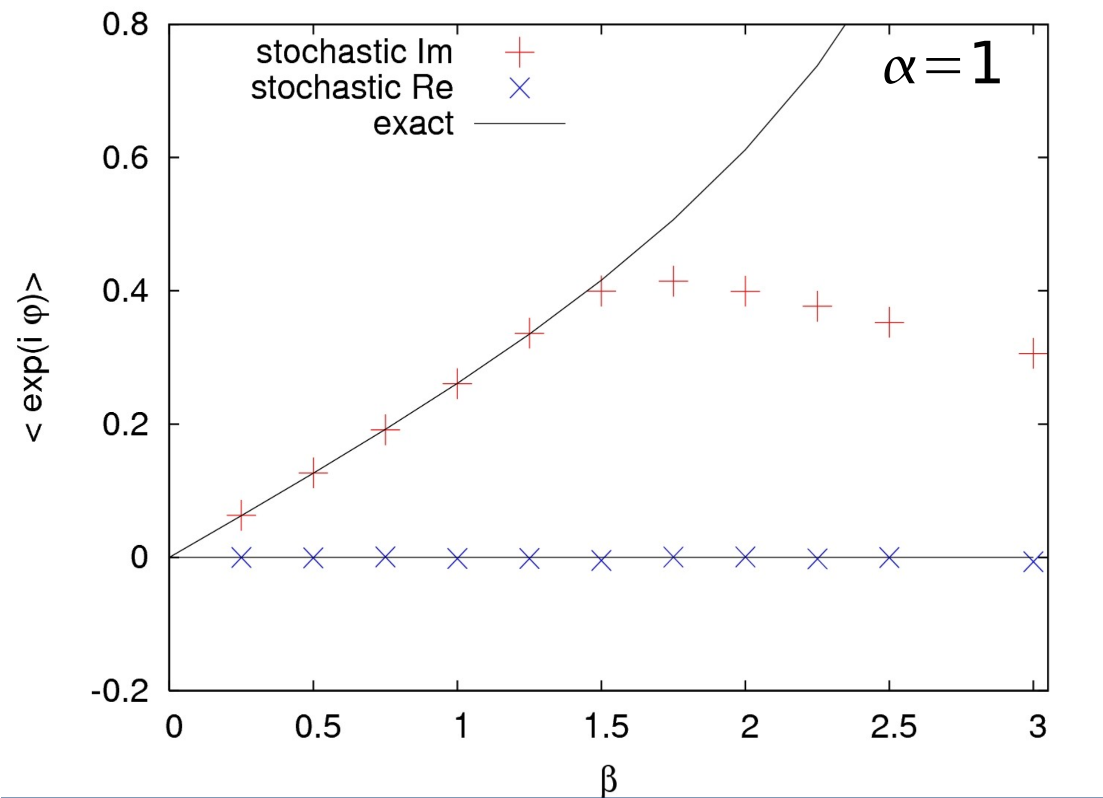


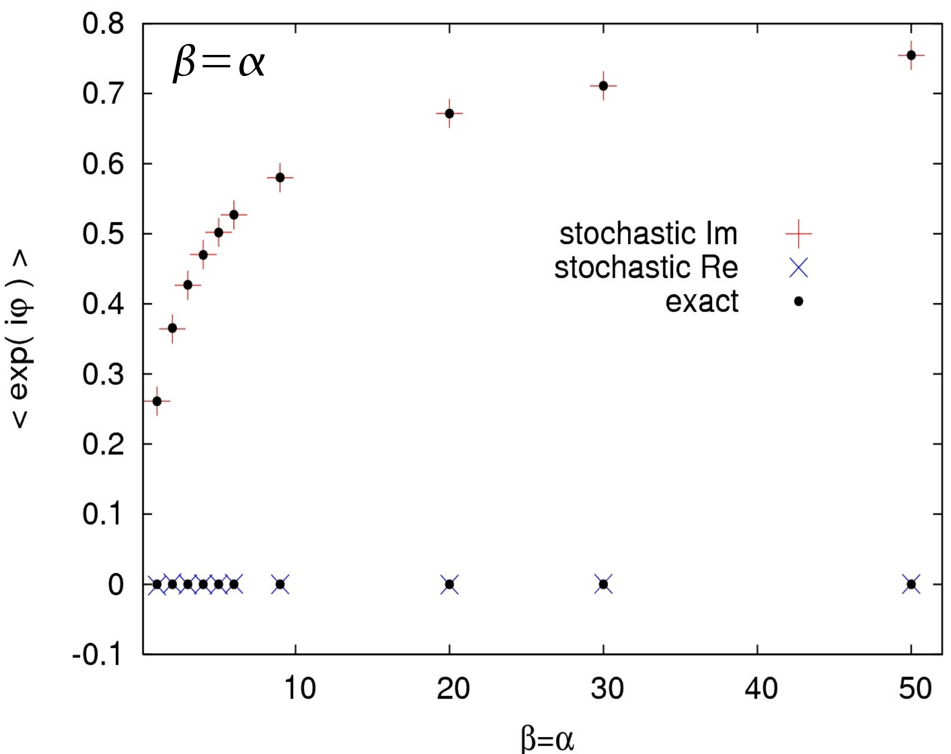
Using the generalized action S_α

Correct results obtained for

$\beta \le \alpha$

With reweighting correct results for S_0





S_α for $\beta = \alpha$

classical fixed point (zero drift term) on the real axis

$\alpha =$ integer

action can be uniqly written as

$S(U) \qquad U \in U(1)$

Correct results for $\langle f(U) \rangle_\alpha \qquad U \in U(1)$