Searching Paths to Understand Systems with a Complex Action

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Sing Problems and Complex Actions
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“Sign Problem” is very tough.

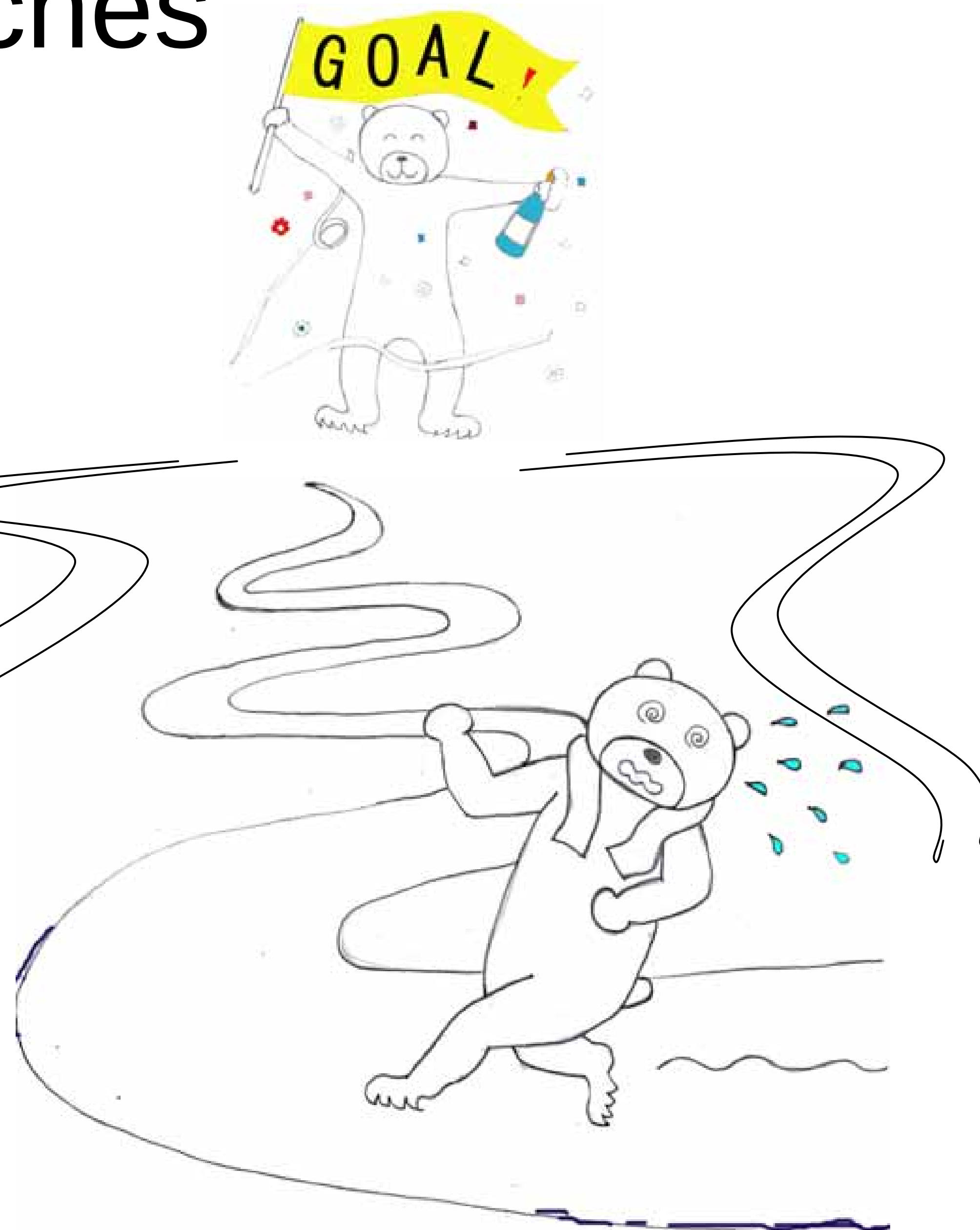
Interesting Problems in Physics suffer from hard sign problem:

QCD at finite density

Real Time Simulations

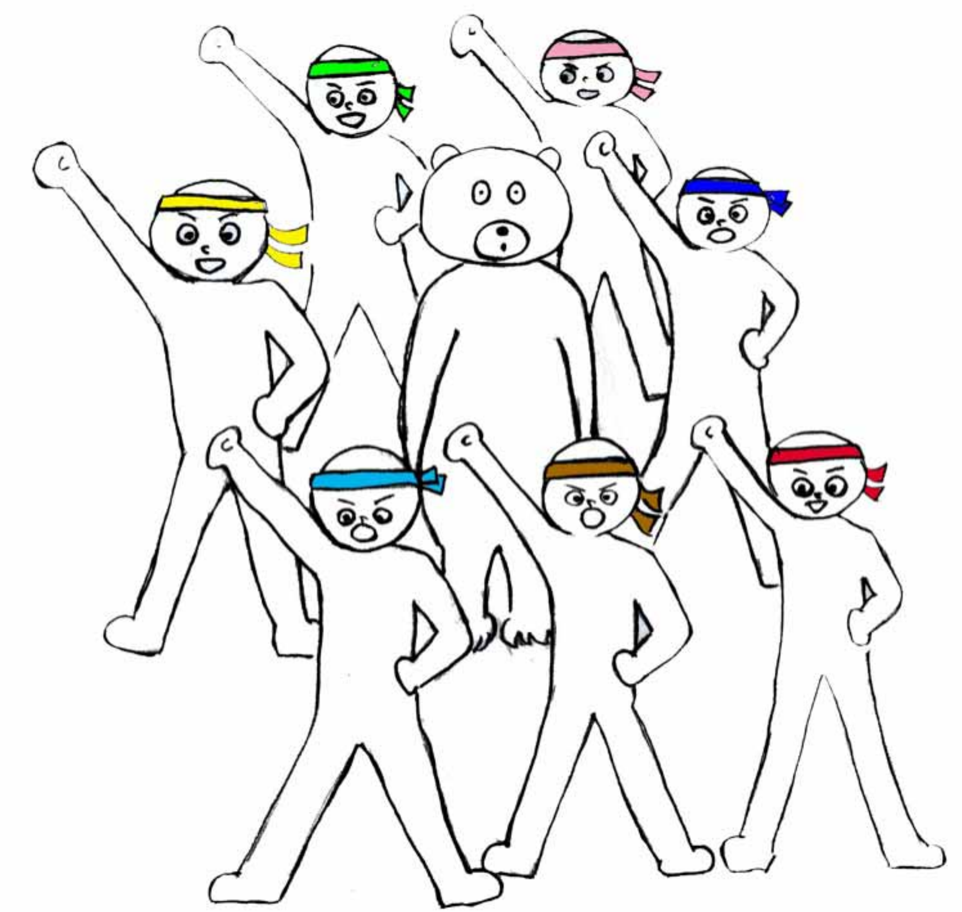
Many Approaches

- Re-Weighting
- Taylor Expansion
- Imaginary μ
- Two-Color
- Random Matrix
- Density of State
- Complex Langevin
- etc.



Many Collaborators

- QCD-TARO
  - deForcrand, Garcia-Perez, Matsufuru, Pushkina, Stamatescu, Takaishi, Umeda
- Kyushu Group
  - Yahiro, Kouno,
- Young
  - Hamada, Suzuki
- Fermion Eigen-Values
  - Akemann, Sasai
- Real Time Simulation
  - Muroya, Mizutani



Two Projects

No New Idea, Simply Doing Standard Things

- Project 1
  - Wilson Fermions with Improved Gauge and Fermion Actions
  - Direct Calculations at Imaginary μ, and Taylor Expansion at Real μ.
    - In Future, Density of State
  - Calculate Hadron Masses, Quark Propagators, Gluon Propagators etc.
- Project 2
  - Real Time Simulation a la Berges & Nucu But for Equilibrium
  - ⇒ Muroya, Lat08@PoS

Some References

- Wilson Fermions with Improved actions
  - WHOT Collaboration (Aoki, Ejiri, Hatsuda, Ichii, Kanaya, Maezawa, Ukita, Umeda)
  - Phys.Rev.D75 (2007) 074501
- Imaginary Chemical Potential with Wilson Fermions
  - Wu, Luo and Chen
  - Phys. Rev. D76 (2007) 034505
  - arXiv:hep-lat/0611035

1. Using Lattice QCD, we are dreaming to investigate large density and low temperature regions some days, where we expect many rich phases of QCD.
2. Low density and High Temperature regions are also very interesting, because they are currently studied experimentally in SPS, RHIC and soon in LHC.
   - ✓ Lattice QCD is expected to provide reliable QCD predictions.
   - ✓ For realistic simulations, we need $N_t \geq 6$ and an improved action.

At finite μ, the fermion determinant $\det D(\mu)$ is complex.

- Why we can not take $\left|\det D(\mu)\right|$ as a measure, and put its phase factor $\exp(i\theta)$ into an observable as reweighting factor ?

$$< O >= \frac{\int DU . O \left|\det \Delta\right| e^{i\theta} \mathrm{e}^{-S_G}}{\int DU \left|\det \Delta\right| \mathrm{e}^{-S_G}} \times \frac{\int DU \left|\det \Delta\right| \mathrm{e}^{-S_G}}{\int DU \left|\det \Delta\right| e^{i\theta} \mathrm{e}^{-S_G}}$$

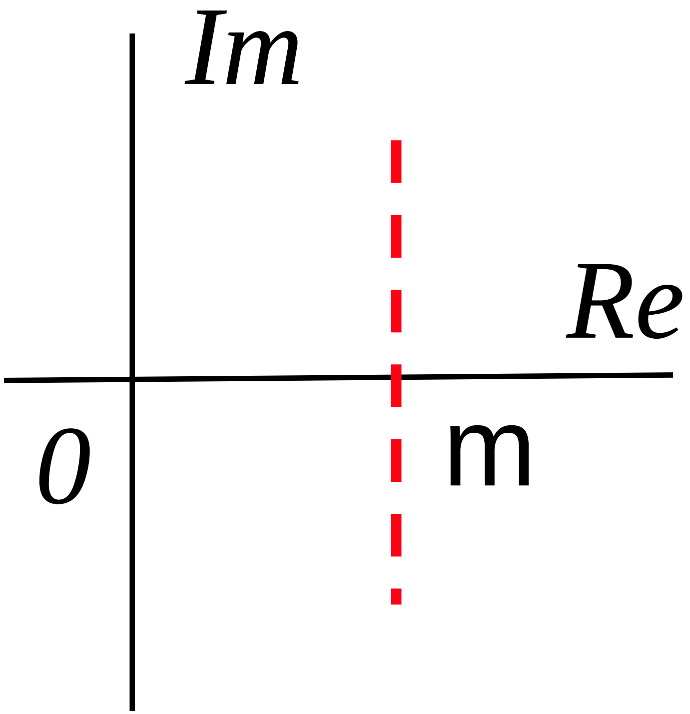
- Because
  - the sign problem
  - eigen values near zero

Difficulty at large Chemical Potential

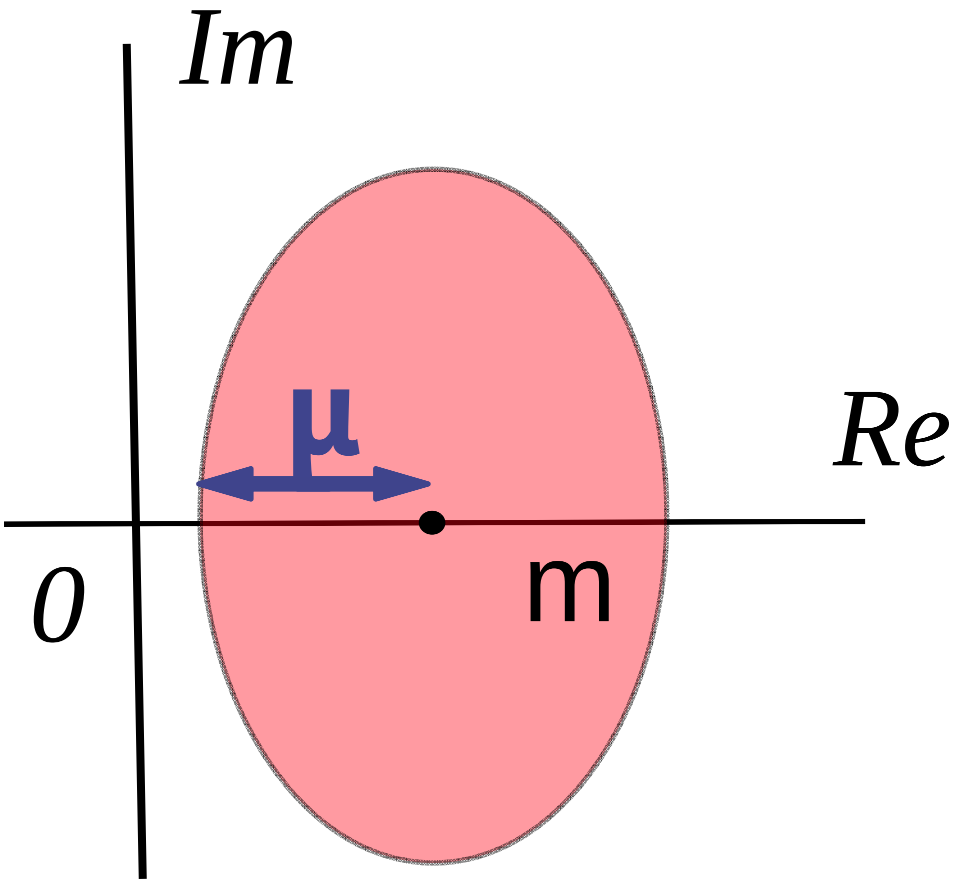
$$\Delta(\mu = 0) = D_\nu \gamma_\nu + m$$

$D_\nu \gamma_\nu$: anti-Hermite

Eigen Value Distribution



When μ increases



$$\frac{|\lambda|_{\max}}{|\lambda|_{\min}} \to \infty$$

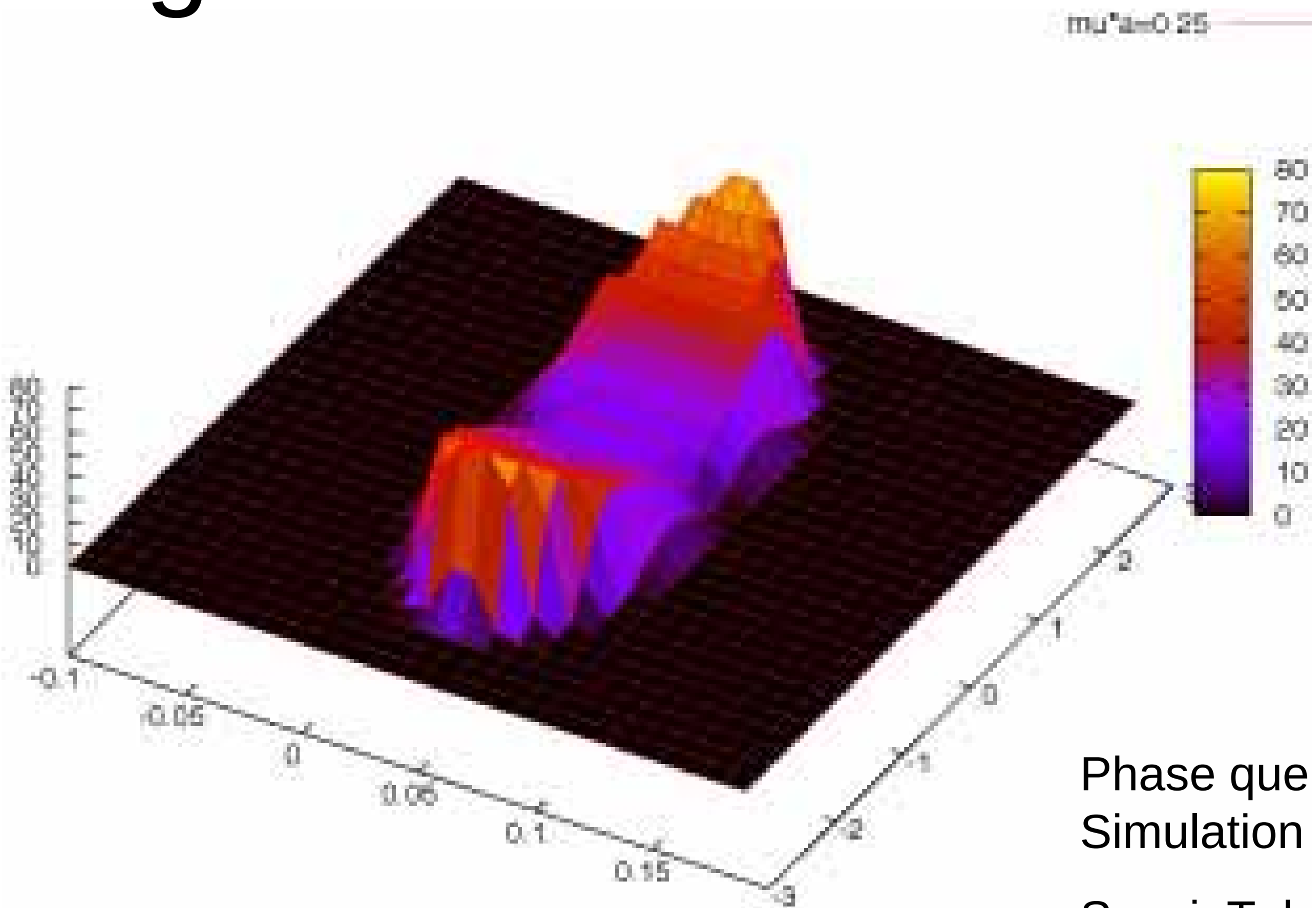
Conjugate Gradient to calculate

$$\Delta(\mu)^{-1}$$

does not converge (Imaginary Chemical Potential formulation does not have this problem.)

Eigen Value Distribution

Eigen Value Distribution

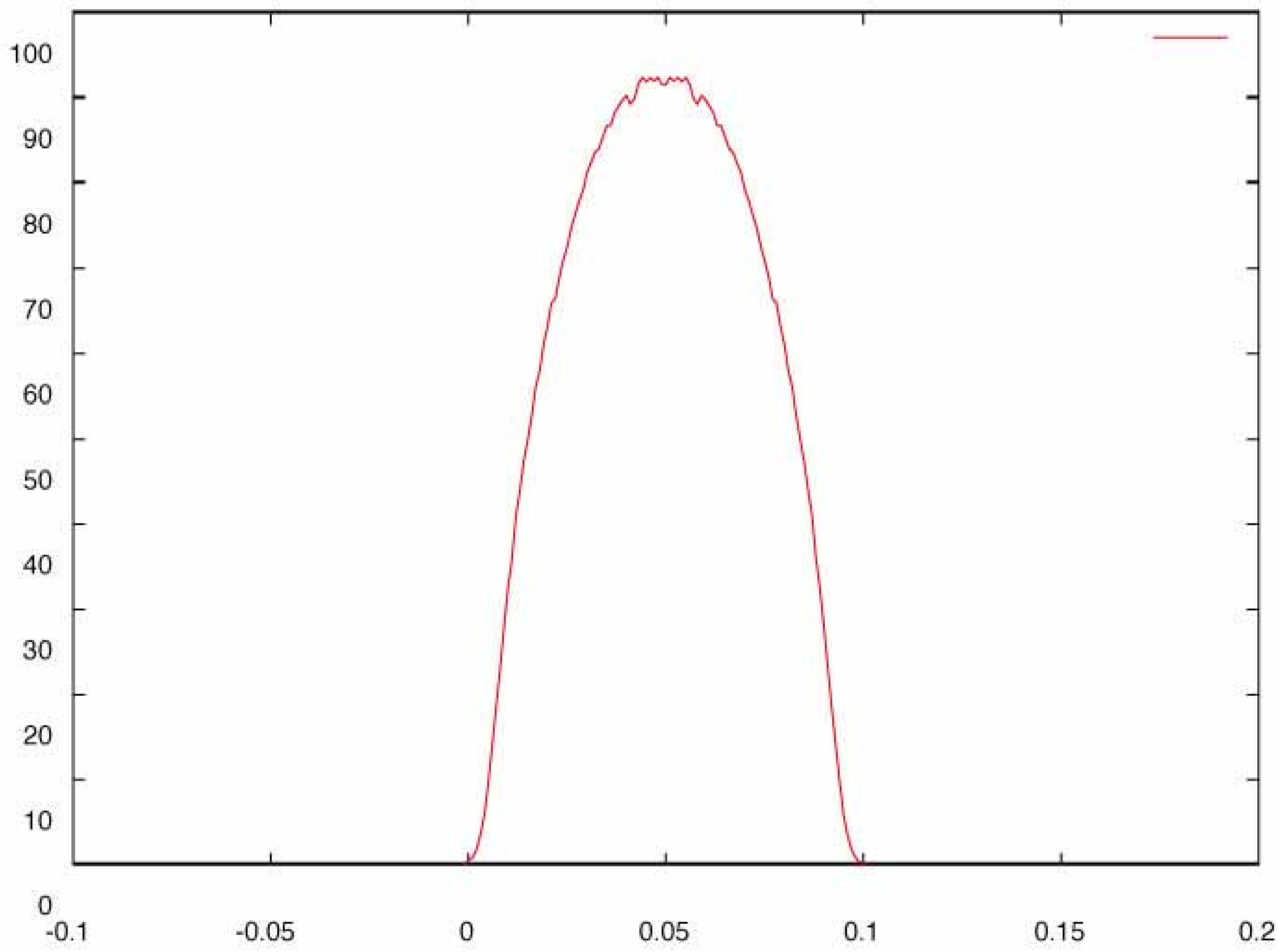


Phase quenching
Simulation

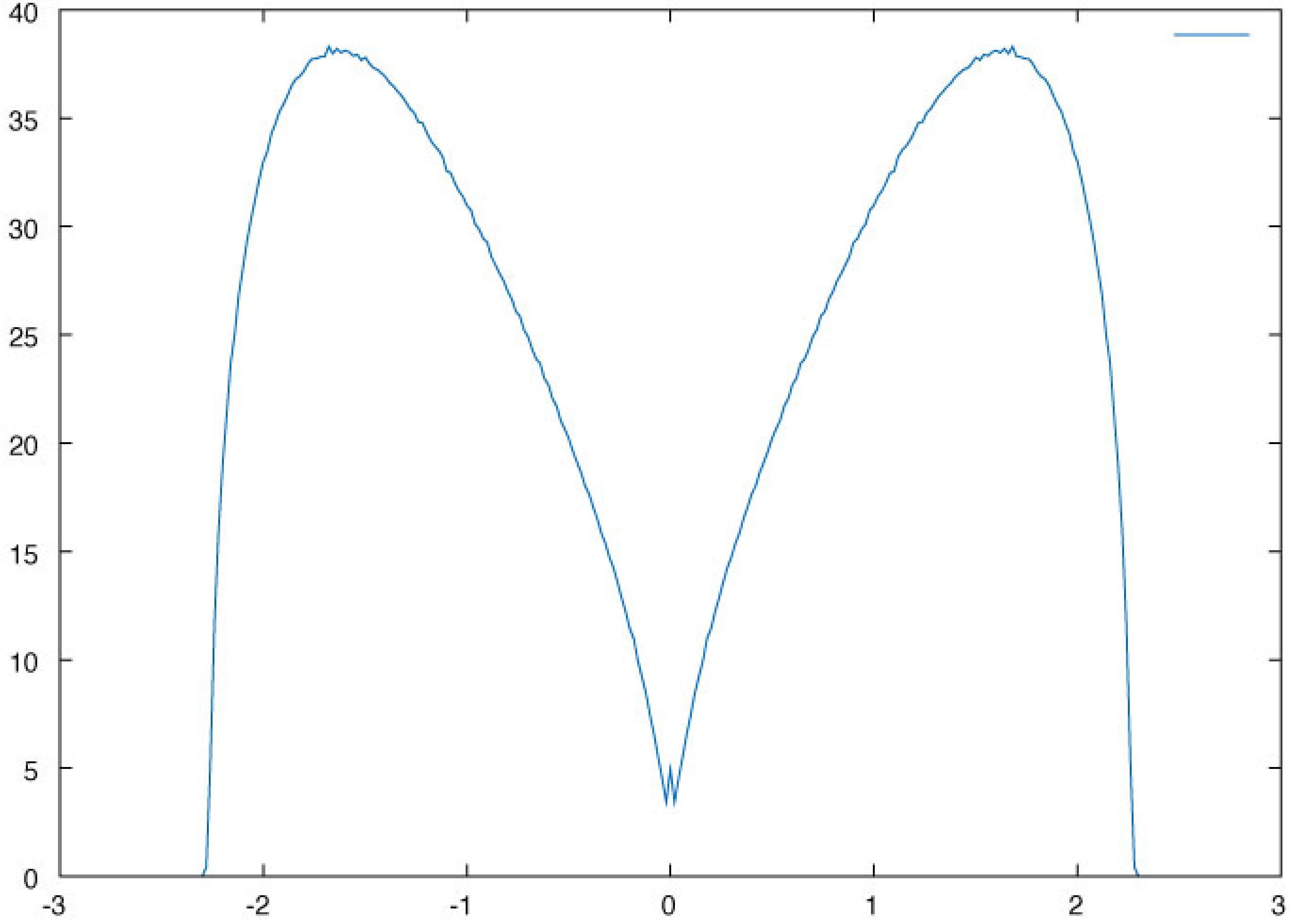
Sasai, Takaishi, A.N.

8^3*4, KS fermions
β=5.3, μa=0.25 (slightly above the transition)

Distribution in *Real* part of μ

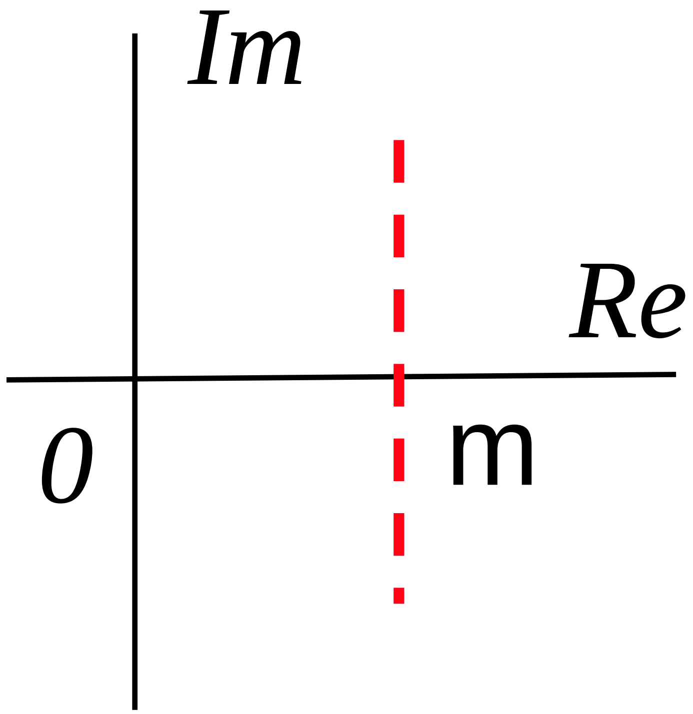


Distribution in *Imag* part of μ



Two Ways to Escape from this Problem, i.e., Without Widening of Fermion Eigen-value Distribution

- Simulate at $\mu=0$ (Fodor-Katz)
- Simulate with Imaginary μ (D'Elia-Lombardo, deForcrand- Philipsen)
  - Fermion Eigen-values remain on the line.



Multi-Parameter Reweighting at Low Chemical Potential

Fodor and Katz

$$\langle O \rangle = \frac{1}{Z}\int DUO \det\Delta(\mu)\,\mathrm{e}^{-S_g(\beta)}$$

$$= \frac{1}{Z}\int DUO\,\mathrm{e}^{-S_g(\beta_0)} \det\Delta(0)\,\mathrm{e}^{S_g(\beta_0)-S_g(\beta)}\,\frac{\det\Delta(\mu)}{\det\Delta(0)}$$

Allton et al.

$$\ln\left(\frac{\det\Delta(\mu)}{\det\Delta(0)}\right) = \sum_{n=1}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n \ln\det\Delta(0)}{\partial\mu^n}$$

- A bit tedious to calculate the higher orders, especially if O is a fermionic observable like screening masses (QCD-TARO).
- Is there an easier way for a lazy person ?
  – possible to write a code in Sunday afternoon.

Fermion Matrix with μ

$$D_{x,x'}(\mu) = \delta_{x,x'} - \kappa\sum_{i=1}^{3}\left\{(1-\gamma_i)U_i(x)\delta_{x',x+\hat{i}} + (1+\gamma_i)U_i^{\dagger}(x')\delta_{x',x-\hat{i}}\right\}$$

$$-\kappa\left\{\xi^{(+)}(x)(1-\gamma_4)U_4(x)\delta_{x',x+\hat{4}} + \xi^{(-)}(x')(1+\gamma_4)U_4^{\dagger}(x')\delta_{x',x-\hat{4}}\right\}$$

$$+c_{SW}\frac{i\kappa}{2}\sum_{\mu,\nu}\sigma_{\mu\nu}P_{\mu\nu}(x)$$

$$\xi^{(+)}(x) \equiv \begin{cases} 1 & 1 \le x_4 < N_4 \\ \exp(+\mu N_4) & x_4 = N_4 \end{cases}$$

$$\xi^{(-)}(x) \equiv \begin{cases} 1 & 1 \le x_4 < N_4 \\ \exp(-\mu N_4) & x_4 = N_4 \end{cases}$$

Here I consider the Wilson + Clover case, but the following argument may be applicable to any fermion action.

(Trivial) Decomposition

$$D(\mu) = D_0 + \Delta D$$

$$D_0 \equiv D(0) \qquad \Delta D \equiv D(\mu) - D(0)$$

$$\Delta D_{x,x'} = \begin{cases} -\kappa(e^{+\mu/T} - 1)(1-\gamma_4)U_4(x)\delta_{x',x+\hat{4}} & (x_4 = N_4, x_4' = 1) \\ -\kappa(e^{-\mu/T} - 1)(1+\gamma_4)U^{\dagger}_4(x')\delta_{x',x-\hat{4}} & (x_4 = 1, x_4' = N_4) \\ 0 & (\text{otherwise}) \end{cases}$$

$$(T = 1/N_4)$$

$$e^{\pm\mu/T}-1=O(\frac{\mu}{T})$$

(Fugacity – 1) expansion ?

Expansion with respect to ΔD

$$\frac{\det D(\mu)}{\det D(0)} = \frac{\det(D_0 + \Delta D)}{\det D_0} = \det\left(I + D_0{}^{-1}\Delta D\right)$$

$$= \exp\left[Tr\log\left(I + D_0{}^{-1}\Delta D\right)\right]$$

$$= \exp\left[-\sum_{l=1}^{\infty}\frac{(-1)^l}{l}Tr\left(D_0{}^{-1}\Delta D\right)^l\right]$$

If *O* includes quark propagators, we use

$$D^{-1} = (D_0 + \Delta D)^{-1} = \sum_{l=0}^{\infty}(-D_0^{-1}\Delta D)^l D_0^{-1}$$

Calculation of $Tr\left(D_0^{-1}\Delta D\right)^l$

- Noise Method

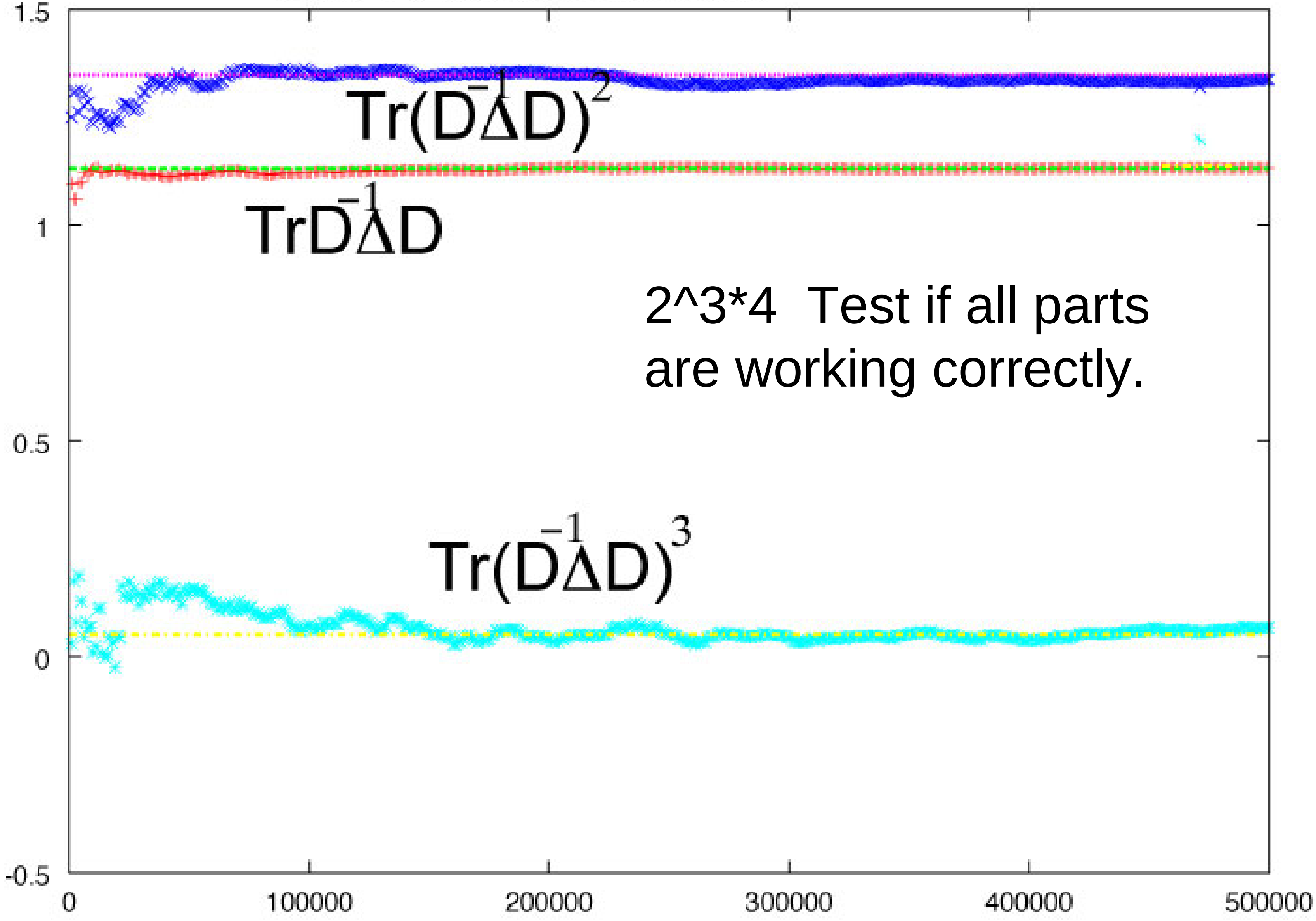
$$D_0\phi = \eta$$

η: Noise

$$\Rightarrow$$

$$\overline{\phi_i \eta_j^*} = \left(D_0^{-1}\right)_{ij}$$

Comparison of Noise Method with Exact result

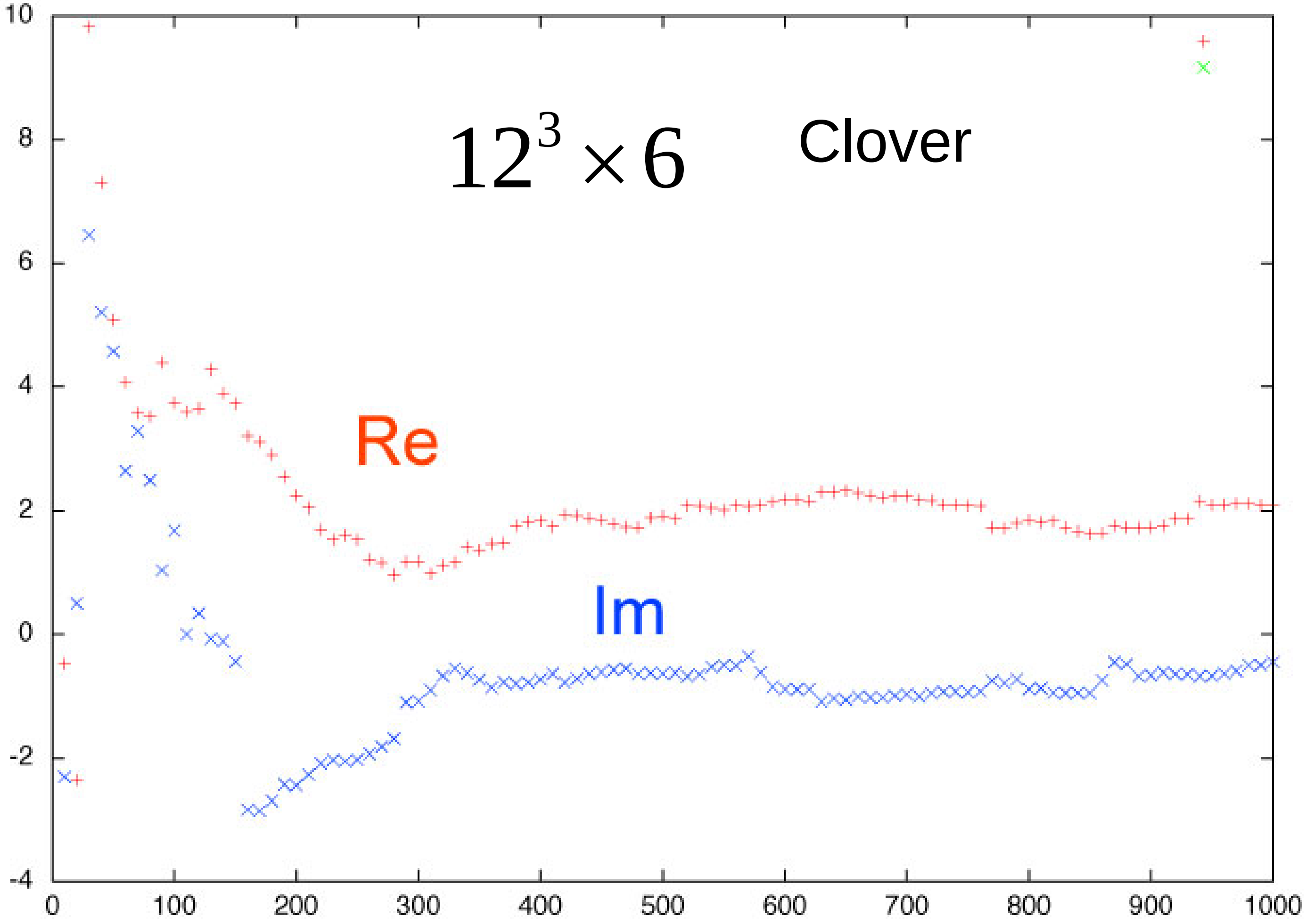


2^3*4 Test if all parts are working correctly.

Very Slow

- We employ a Technique by Foley et al
  - Dilution
  - Hybrid-List
  - See Nakamura, Lat08@PoS

$$\frac{\det D(\mu)}{\det D(0)} \simeq \exp\left[-\sum_{l=1}^{4} \frac{(-1)^l}{l} Tr\left(D_0{}^{-1}\Delta D\right)^l\right]$$



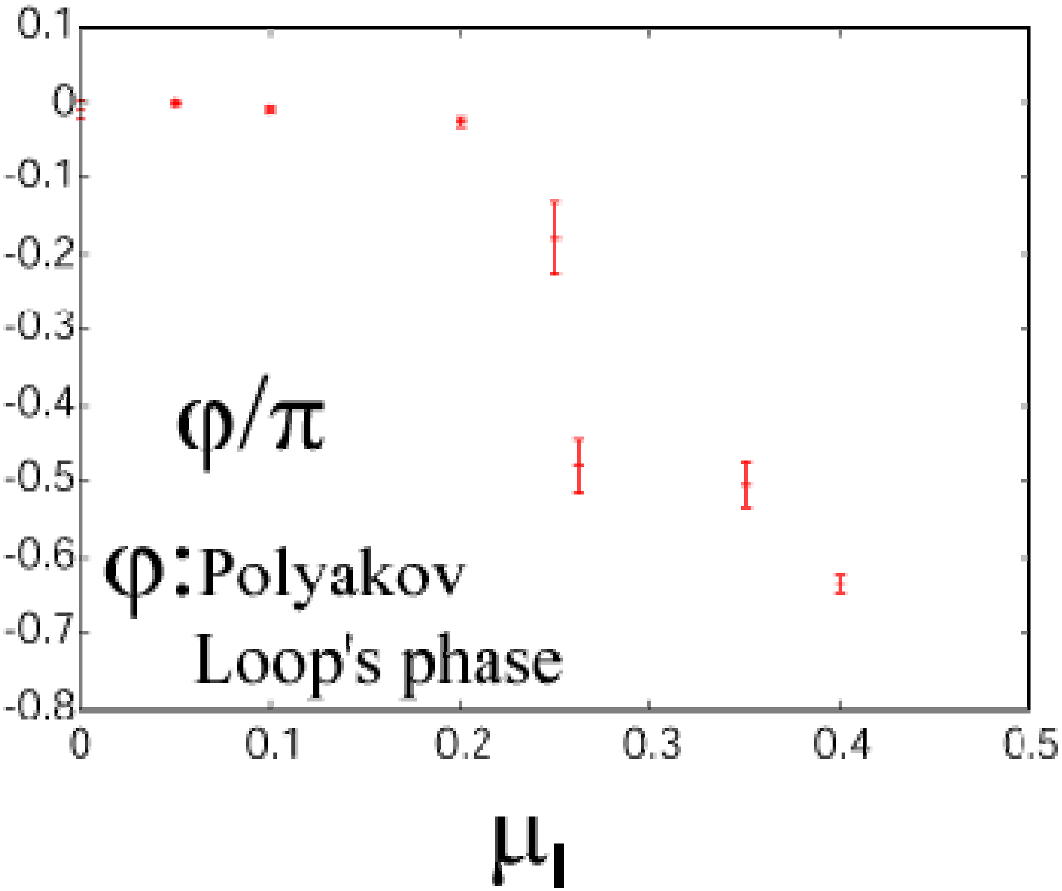
Imaginary Chemical Potential

- Wilson Fermions with Clover + Iwasaki Gauge action

8*8*8*4

β=1.90, κ=1.38817

T/Tc=1.08



PNJL model (Kyushu Group)

$$M(\theta) = \sum_{k=0} a_k \cos(3k\theta)$$

$$\mu_I = T\theta$$

$$M(\mu_R / T) = \sum_{k=0} a_k \cosh(3k\mu_R / T)$$

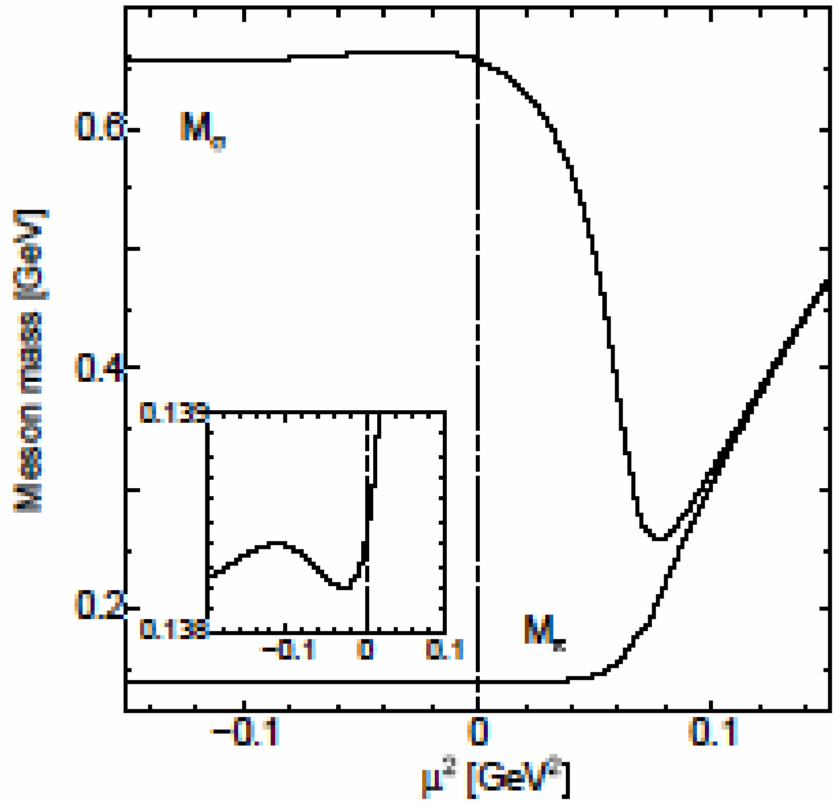


Fig. 1: The μ^2-dependence of sigma and pi meson masses, M_σ and M_π, at $T = 160$ MeV. The inset represents the μ^2-dependence of pion mass near $\mu^2 = 0$.

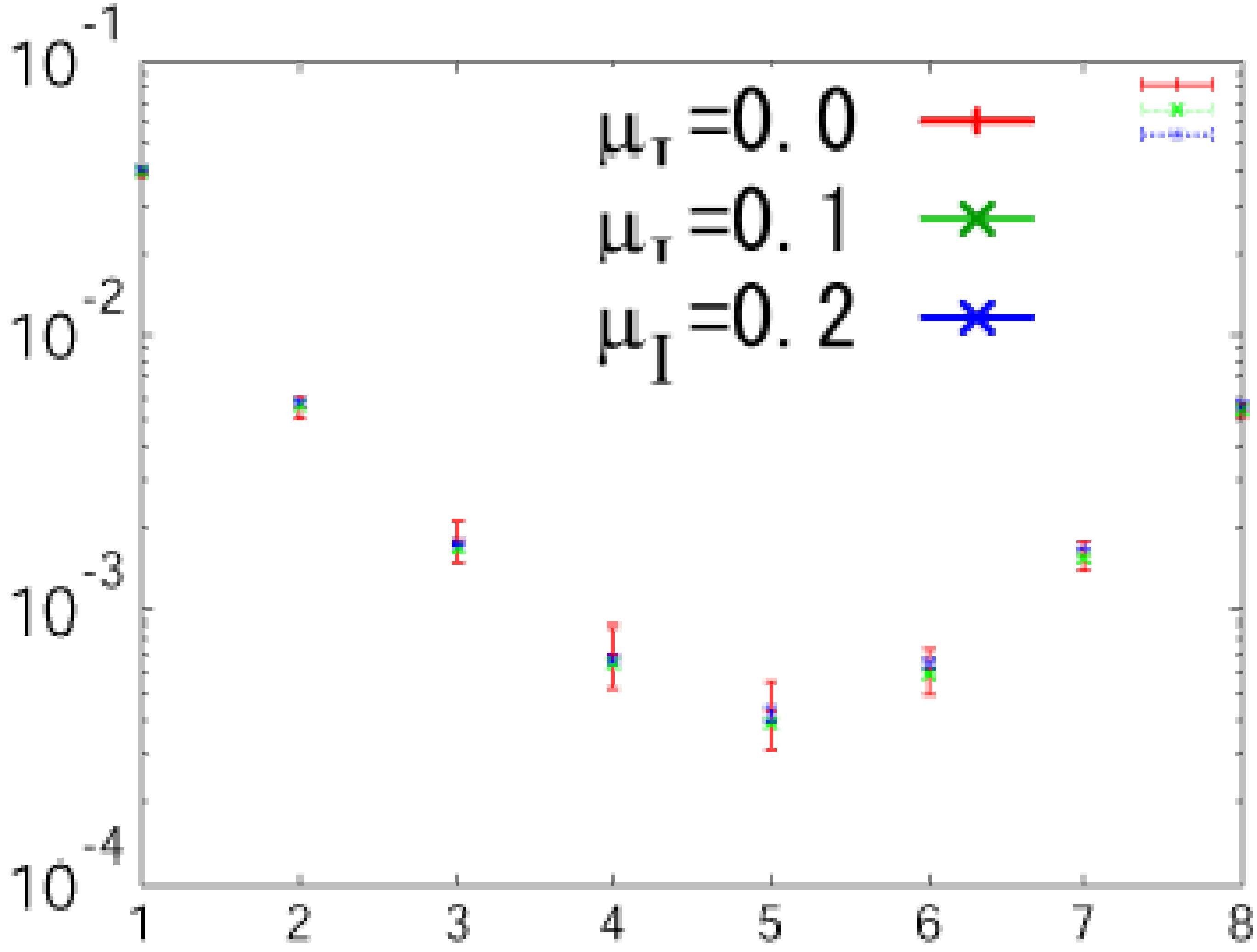
Lattice Data (Preliminary)

8*8*8*4

β=1.90,

κ=1.38817

T/Tc=0.504



Project 2

Do you know Very Interesting Papers by Berges-Stamatescu ?

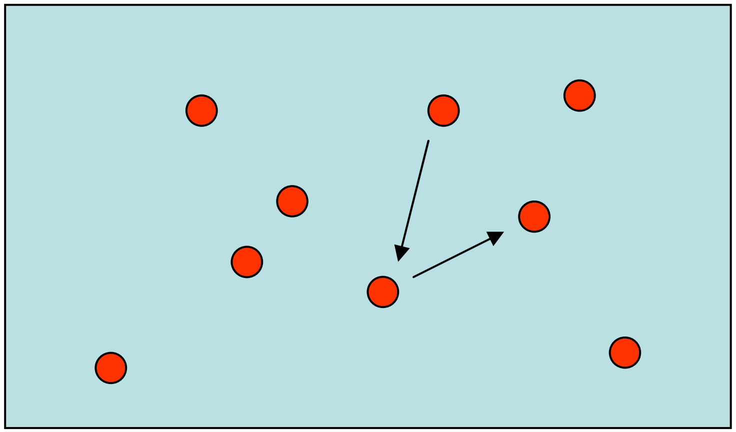
Your Old UnKnown Papers are also Cited.

Berges-Stamatescu, Phys.Rev.Lett. 95 (2005) 202003.

Berges, Borsanyi, Sexty & Stamatescu, Phys.Rev.D75 (2007) 045007.

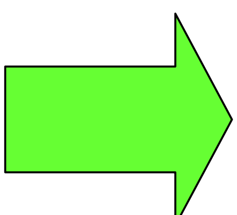
Real Time Simulation is finally accomplished !!!

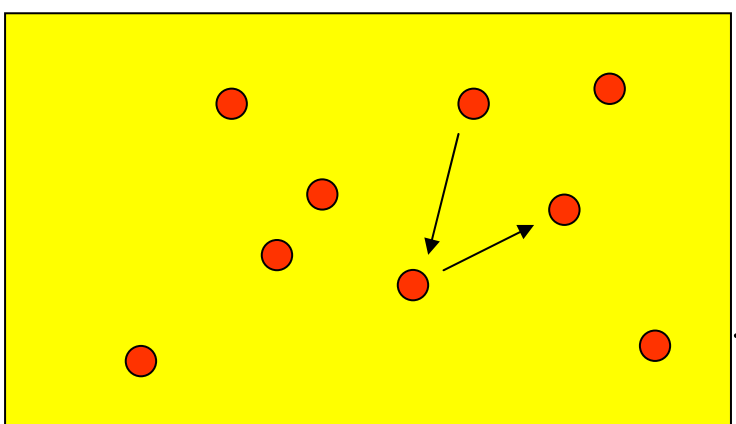
- Monte Carlo Simulation



$$P = e^{-S}$$

- Complex Langevin





$$\frac{d\phi}{d\tau} = -\frac{\delta S}{\delta \phi} + (\text{Noise})$$

Stochastic Quantization

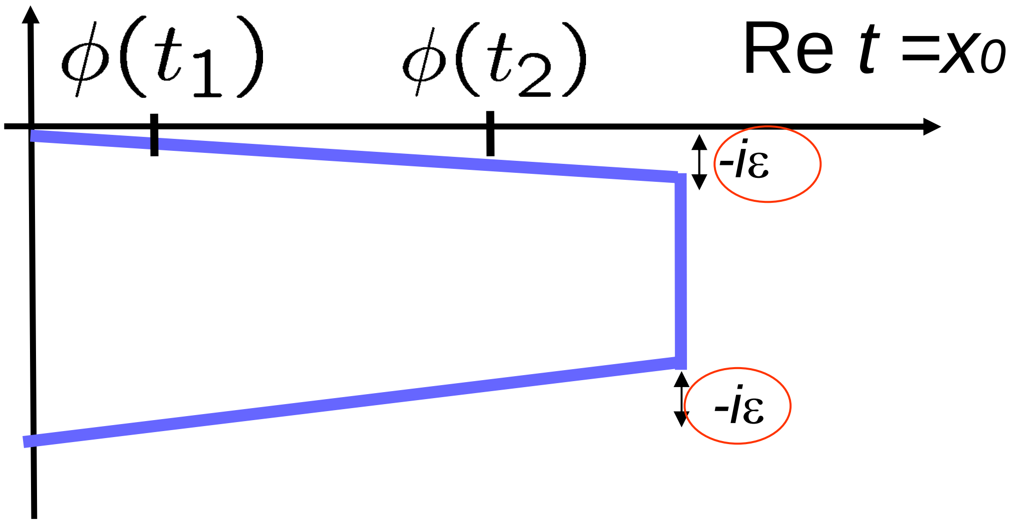
$$\frac{d\phi}{d\tau} = -\frac{\delta S}{\delta \phi} + (\mathrm{Noise})$$

- Parisi-Wu
  - Equivalent to Standard Quantization
  - Langevin Eq. (τ : Monte Carlo Time)
- Parisi : Complex Langevin
  - Monte Carlo Simulation is impossible for Complex S, But Langevin works !
  - Wrong Solution Problem:
    - Ambjorn-Yang, Matsui-Nakamura, Okano, Klauder

Berges et al. Schwinger-Dyson Idensities for n-Point Function are checked numerically.

Difficulty of Minkowski (1)

- e^{iS_M} Measure is now well defined.
  - We define in Euclidean, and come back to Minkowski by the Analytic Continuation.
  - How possible in Numerical Simulation
- Schwinger -Keldysh type closed time path in Numerical Simulations



Free Field Case

$$L = \frac{1}{2}\left[\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi\right]$$

$$Z = \int \prod d\varphi^* d\varphi e^{iS}$$

$$= \int \prod d\varphi(p)^* d\varphi(p) e^{-\sum_p \varphi(p)^* \{\cdots\} \varphi(p)}$$

Real Part of $\{\cdots\}$

$$= \varepsilon \left\{ \sum_\mu (1 - \cos p_\mu) + \frac{1}{2} m^2 \right\}$$

ε is neces-sary

Difficulty in Minkowski (2)

$$\frac{\partial^2}{\partial x_1^{\ 2}}+\frac{\partial^2}{\partial x_2^{\ 2}}+\frac{\partial^2}{\partial x_3^{\ 2}}+\frac{\partial^2}{\partial x_4^{\ 2}} \Rightarrow \frac{\partial^2}{\partial x_1^{\ 2}}+\frac{\partial^2}{\partial x_2^{\ 2}}+\frac{\partial^2}{\partial x_3^{\ 2}}-\frac{\partial^2}{\partial t^2}$$

Poisson Boundary Problem

Cauchy Initial Value Problem

Instability

Berges-Stamatescu impose a boundary condition

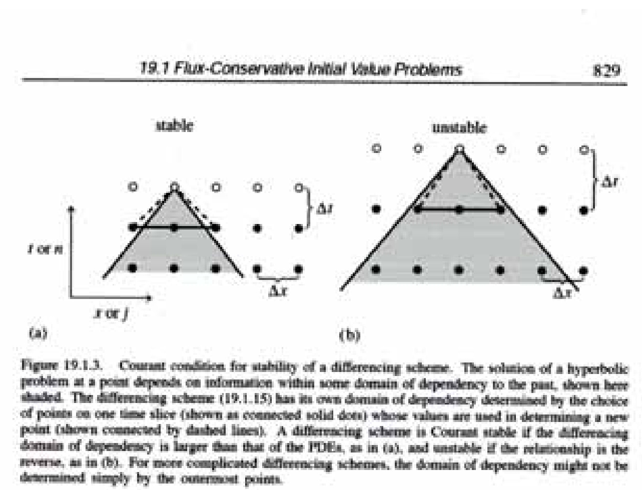

19.1 Flux-Conservative Initial Value Problems 829

Figure 19.1.3. Courant condition for stability of a differencing scheme. The solution of a hyperbolic problem at a point depends on information within some domain of dependency to the past, shown here shaded. The differencing scheme (19.1.15) has its own domain of dependency determined by the choice of points on one time slice (shown as connected solid dots) whose values are used in determining a new point (shown connected by dashed lines). A differencing scheme is Courant stable if the differencing domain of dependency is larger than that of the PDEs, as in (a), and unstable if the relationship is the reverse, as in (b). For more complicated differencing schemes, the domain of dependency might not be determined simply by the outermost points.

Numerical Recipe

Courant Condition

$$\frac{|v|\Delta t}{\Delta x}\leq 1$$

d: Spatial Dimension

$$\frac{|v|\Delta t}{\Delta x}\leq \frac{1}{\sqrt{d}}$$

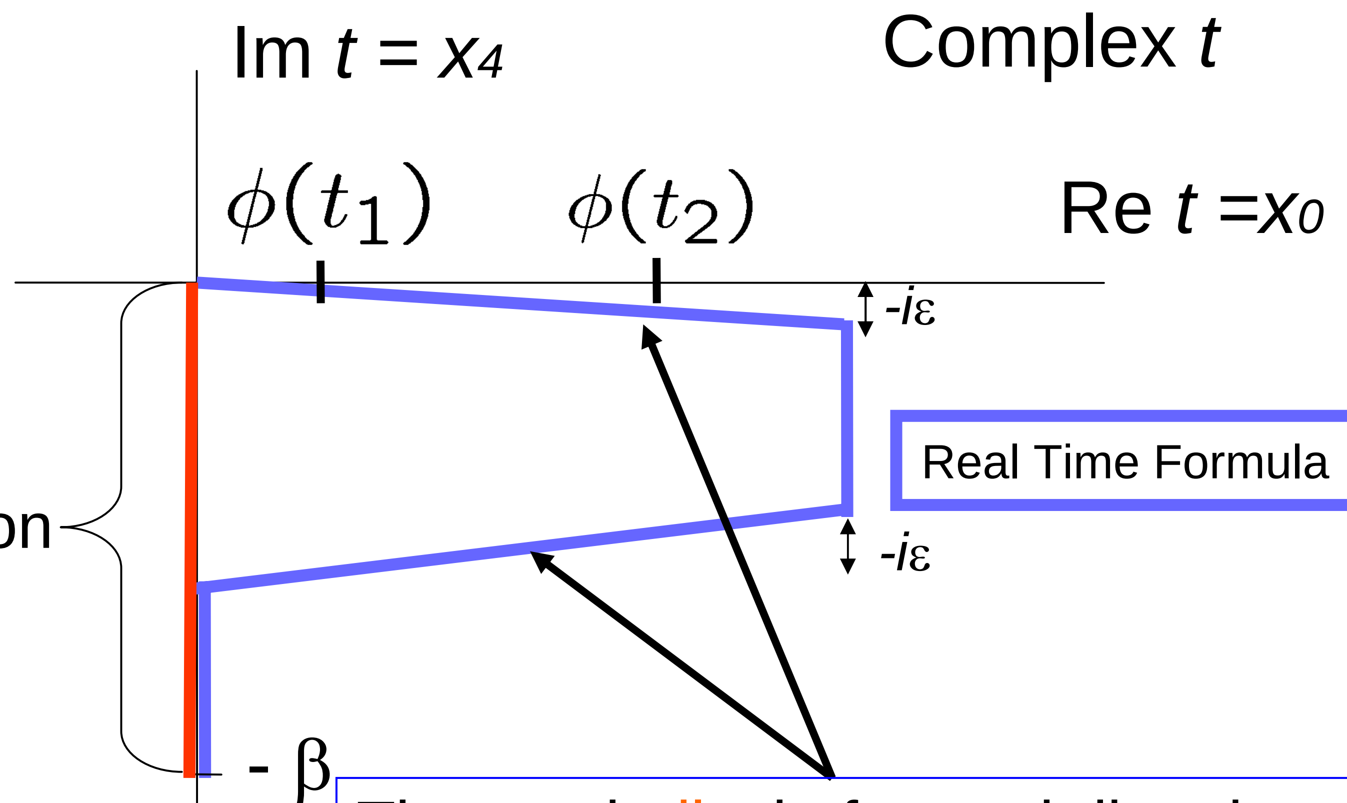
$$\Rightarrow \Delta t \leq \Delta x$$

An-Isotropic Lattice

Here, what we want to do

- Berges-Stamatescu ⇒Non-Equilibrium
  - Great ! This is a Dream of Physicists.
- But even within the Equilibrium Process, This is a great possibility:
  - Lattice QCD：First-Principle Calculation
  - But Analytic Continuation is difficult in Numerical Simulations:
    - Matsubara Green Function ⇒ Spectral Function ⇒ Advanced/Retarded Green Function
- Test in Scalar Field Theory

Our plan



(Anti-)Periodic Boundary Condition

⇕

KMS condition
Thermal equilibrium
(by Niegawa)

Time path tilts in forward direction
(Feynman causality,
Hamiltonian has spectra bounded from below)

Imaginary Time Formula

$$e^{iS_M} \to e^{-S_E}$$

Wick rotation

Small $-i\varepsilon$ in *real time path*.

⇕ Nakazato and Yamanaka

Converge stochastic process in SQM

⇐

$$e^{iS_M}$$

Quantum Mechanics

$$Z = \int \Pi dx(t) e^{iS}$$

$$S = \int dt L \qquad L = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)$$

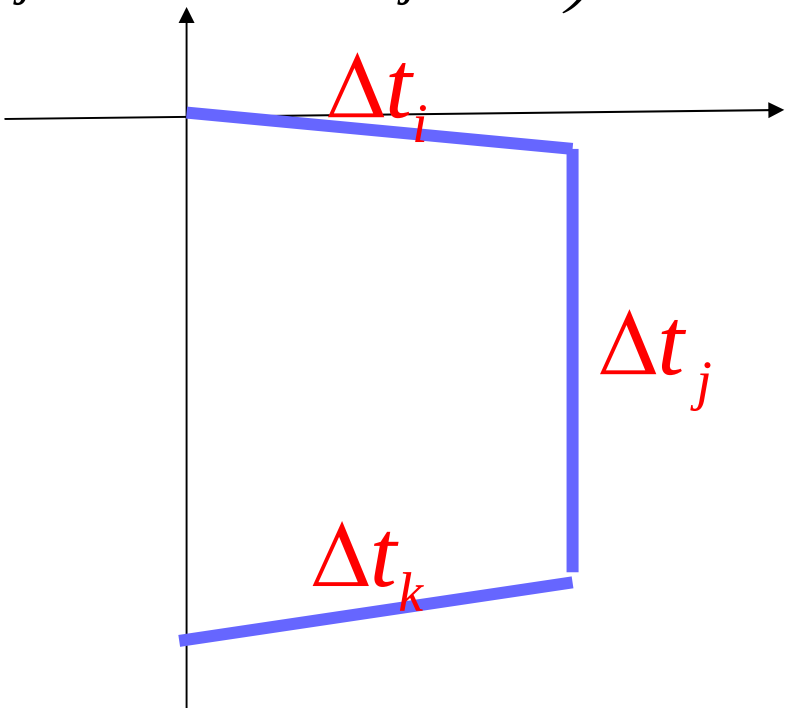
$$\frac{dx}{d\tau} = \frac{i\partial S}{\partial x} + (\text{Noise})$$

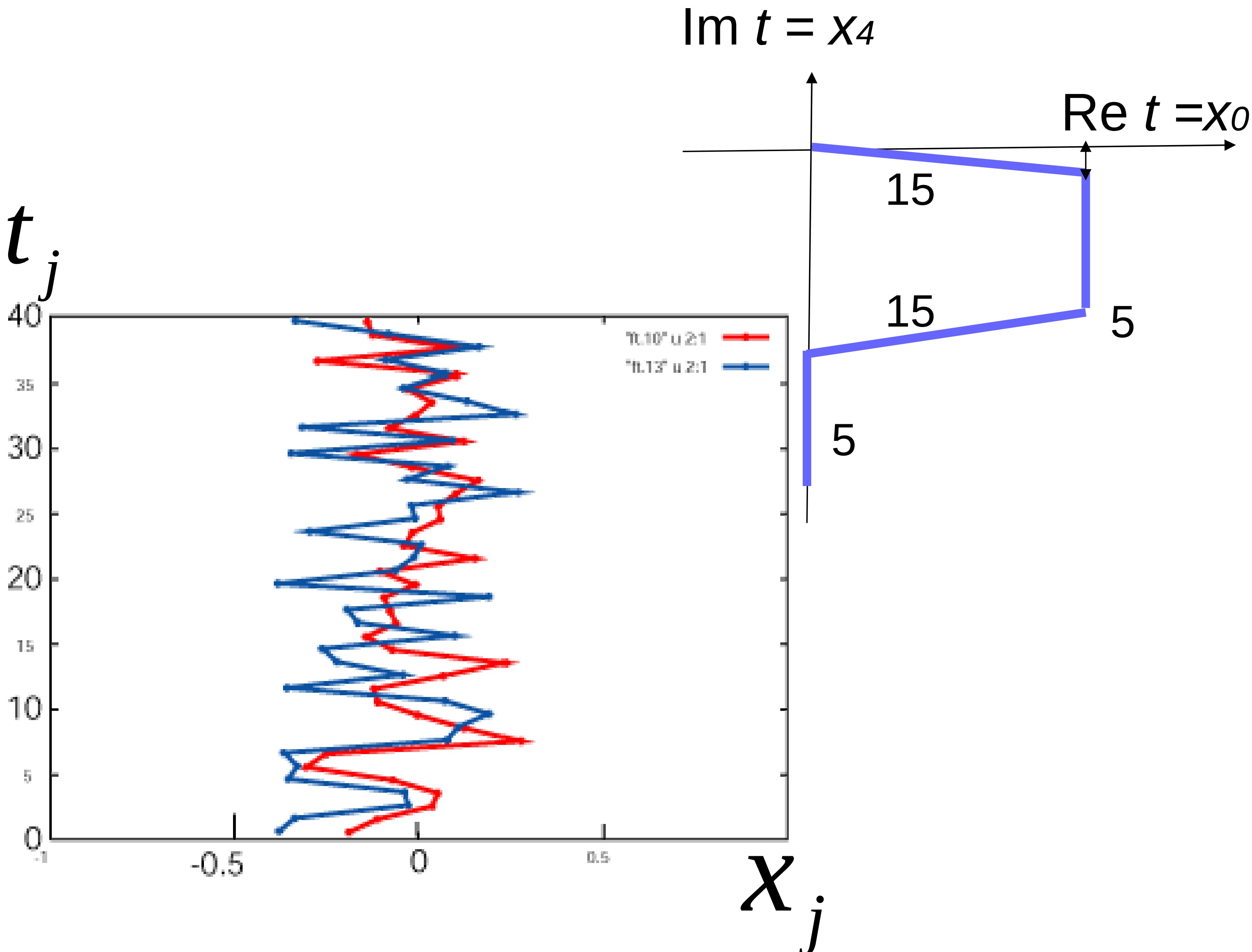
$$S = \sum_j \Delta t_j \left\{ -\frac{m}{2} x_j \nabla_t^2 x_j - V(x_j) \right\}$$

$$\nabla_t^2 x_j = \frac{1}{\Delta t_j} \left(\frac{x_{j+1} - x_j}{\Delta t_j} - \frac{x_j - x_{j-1}}{\Delta t_{j-1}} \right)$$

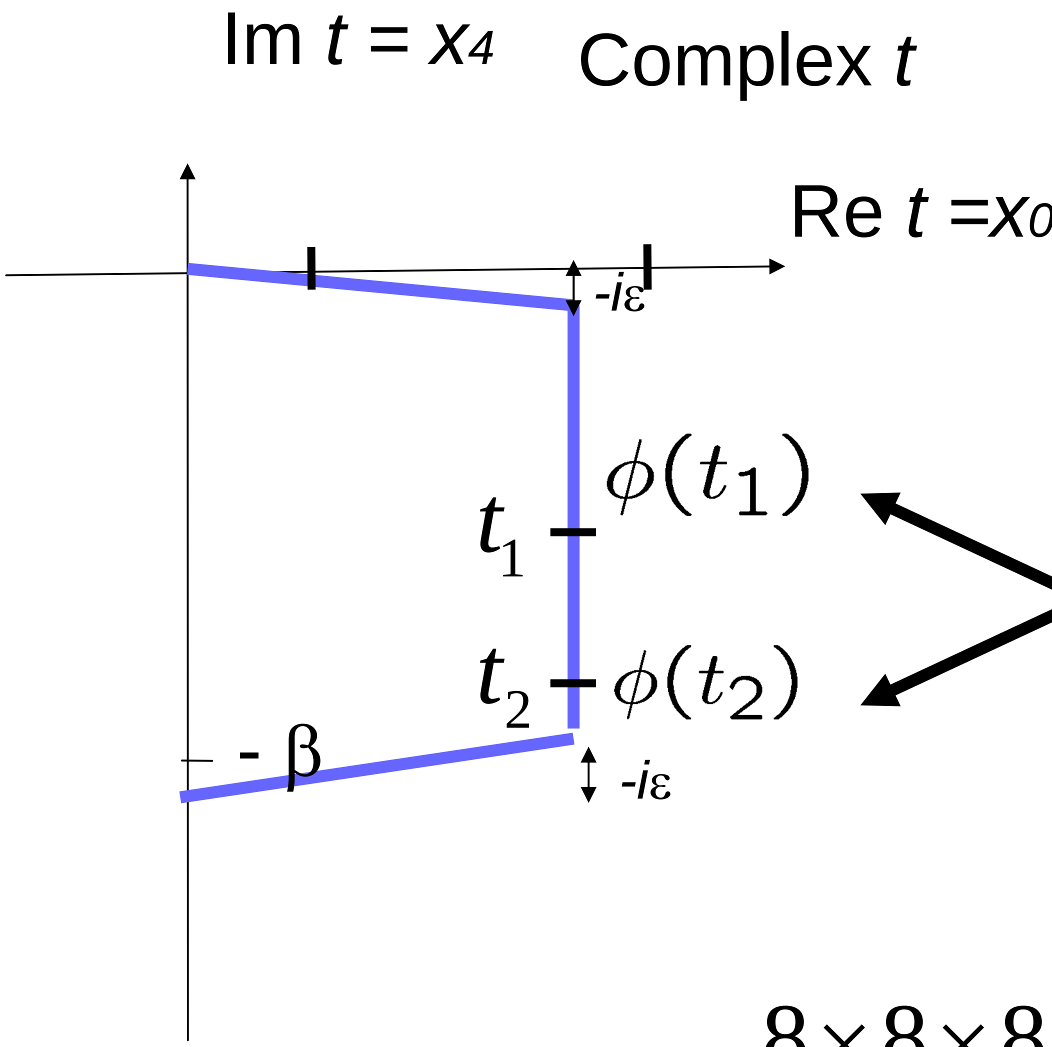
$$\Delta t_j \equiv t_j - t_{j-1}$$

Complex now !





Im t = x_4
Re t =x_0
15
15
5
5
t_j
40
35
30
25
20
15
10
5
0
-1
-0.5
0
0.5
X_j

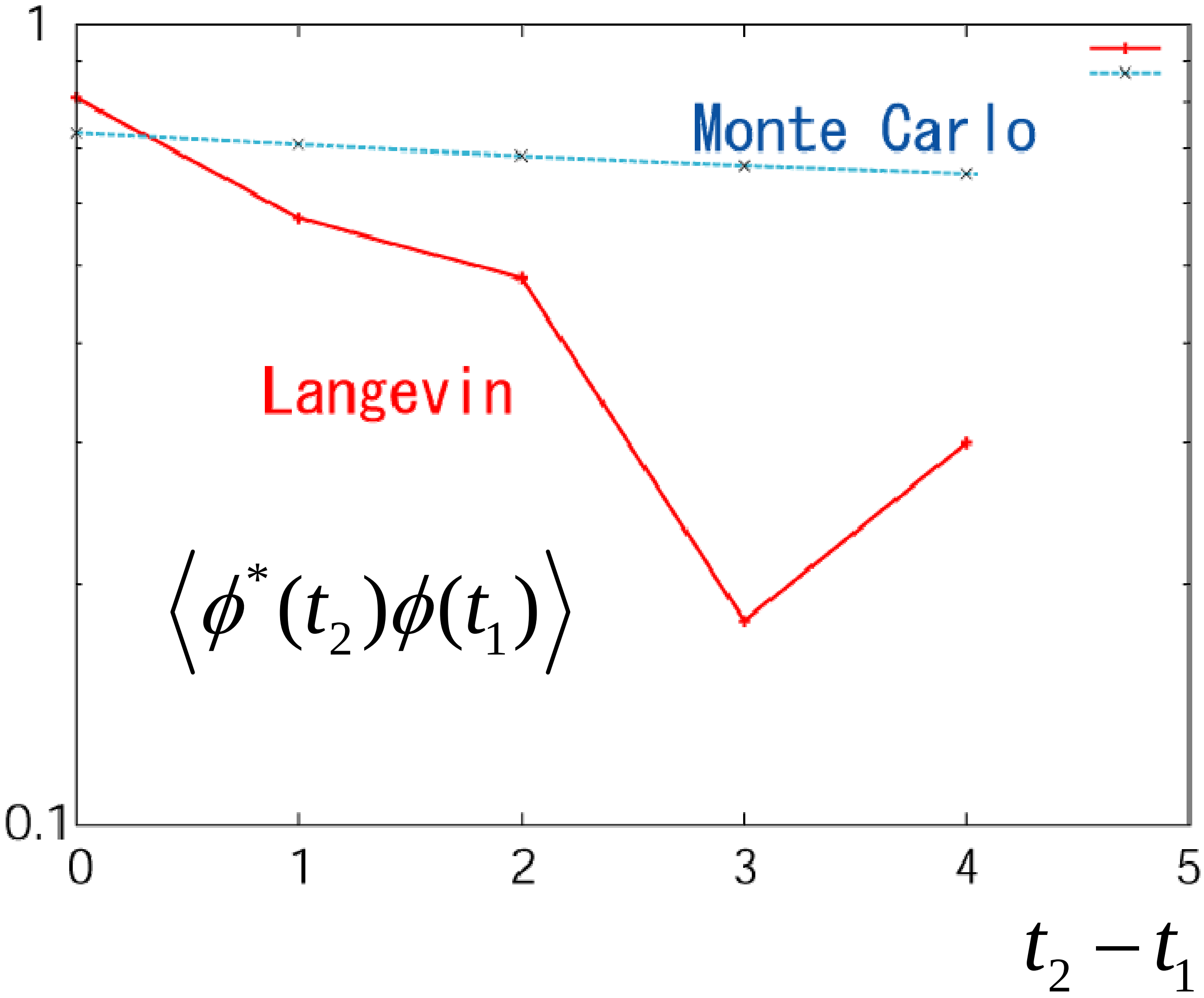


Here we expect the same result as Ordinal (Imaginary Time) Path Integral.

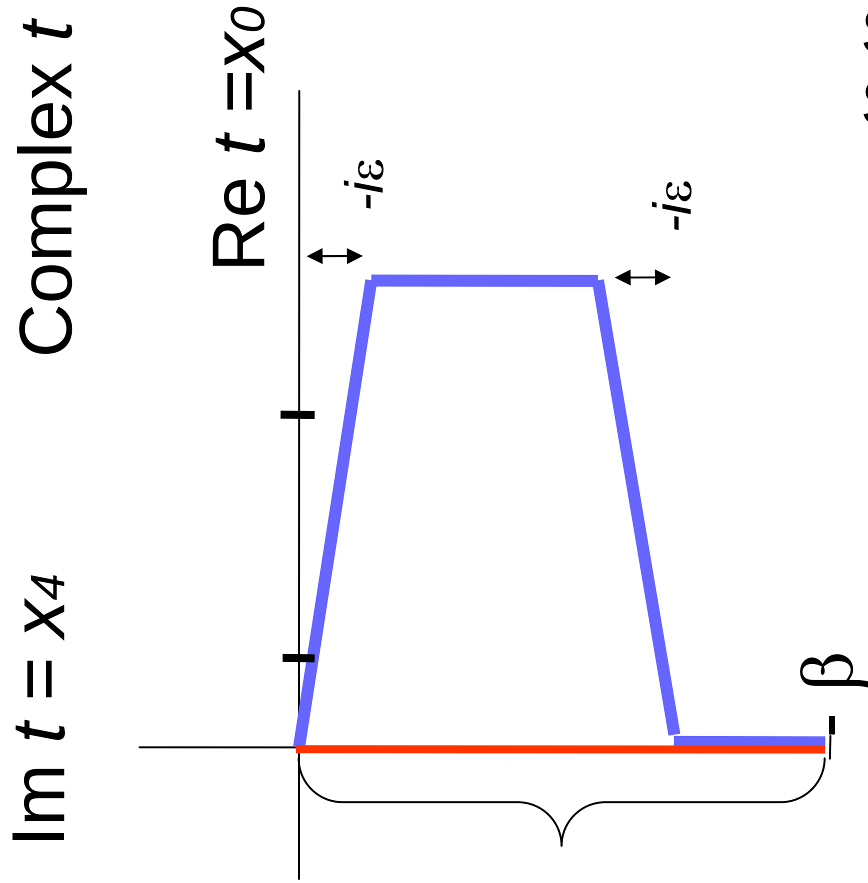
$$8\times8\times8\times(20=2+16+2)$$

$$m=0.2,\lambda=0.05$$

Very Preliminary



t_1: Midpoint in the imaginary path.



(Anti-)Periodic Condition

Stochastic Quantization
Starts from Euclidean Action

$$S_E$$

We extend
Euclidian action to
complex plane

$$\frac{d\phi}{d\tau} = -\frac{\delta S_E}{\delta \phi(\tau)} + \eta(\tau)$$

$$\frac{\delta S_E}{\delta \phi(t_C, \boldsymbol{x}, \tau)} = \Delta t_c \; [\partial_{t_c}\partial_{t_c}\phi(t_c, \boldsymbol{x}, \tau) \cdots$$

Contour dependent phase

Numerical simulation

- Scalar field $\lambda|\phi|^4$

  $ma = 0.2, \quad \lambda = 0.01, 0.05, \ 0.1$

- Lattice size

  16 X 16 X 16 X 40, tilt = 0.05

  40= 15 + 5 + 15 + 5,

  15: real, 5: imaginary, 15: real, 5: imaginary

- Stochastic process $\Delta\tau$=0.00002

  Take average for each 5000 steps X 50times

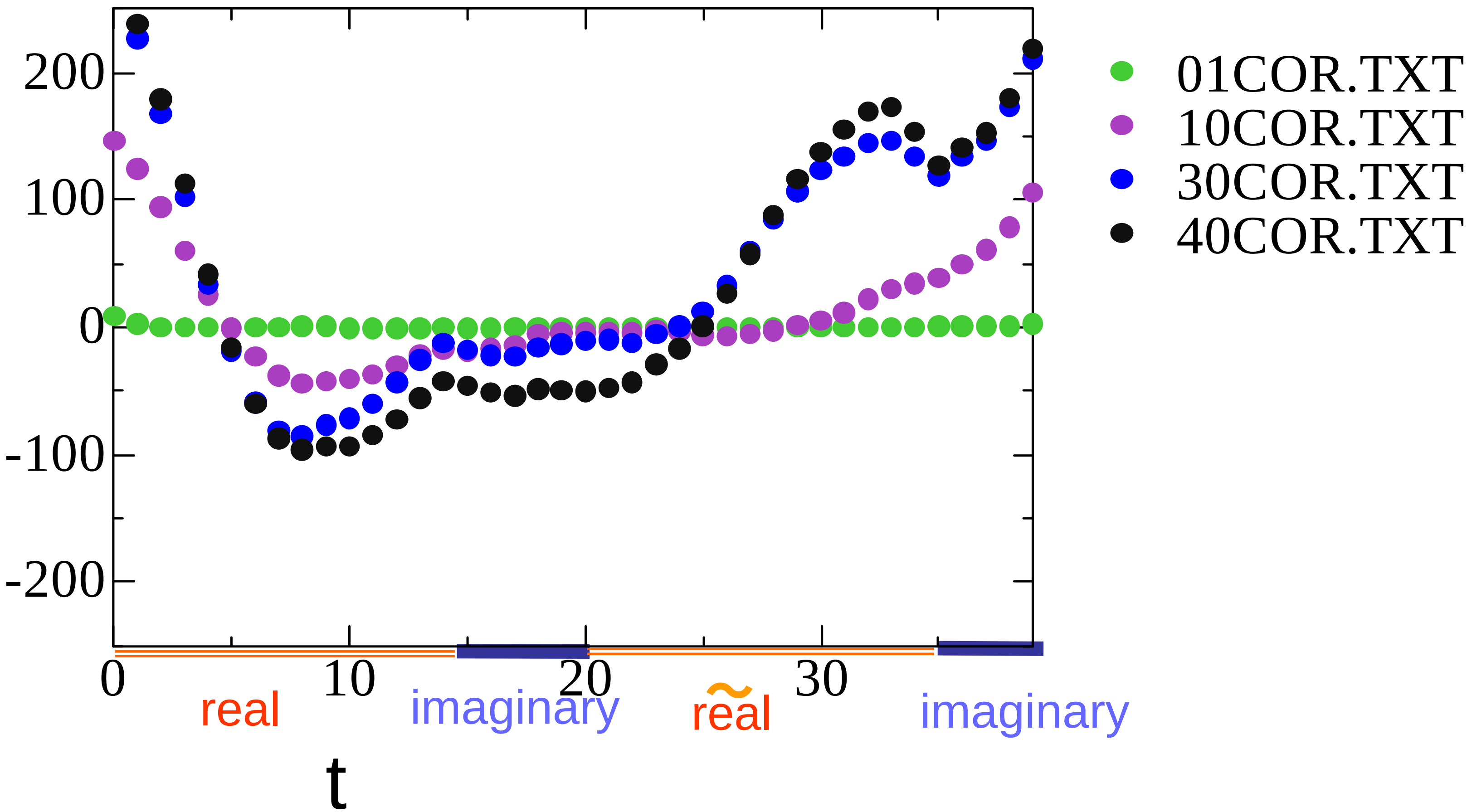
- Anisotropic

  spatial lattice size

  = time-like lattice size × γ, γ=4

  Courant condition

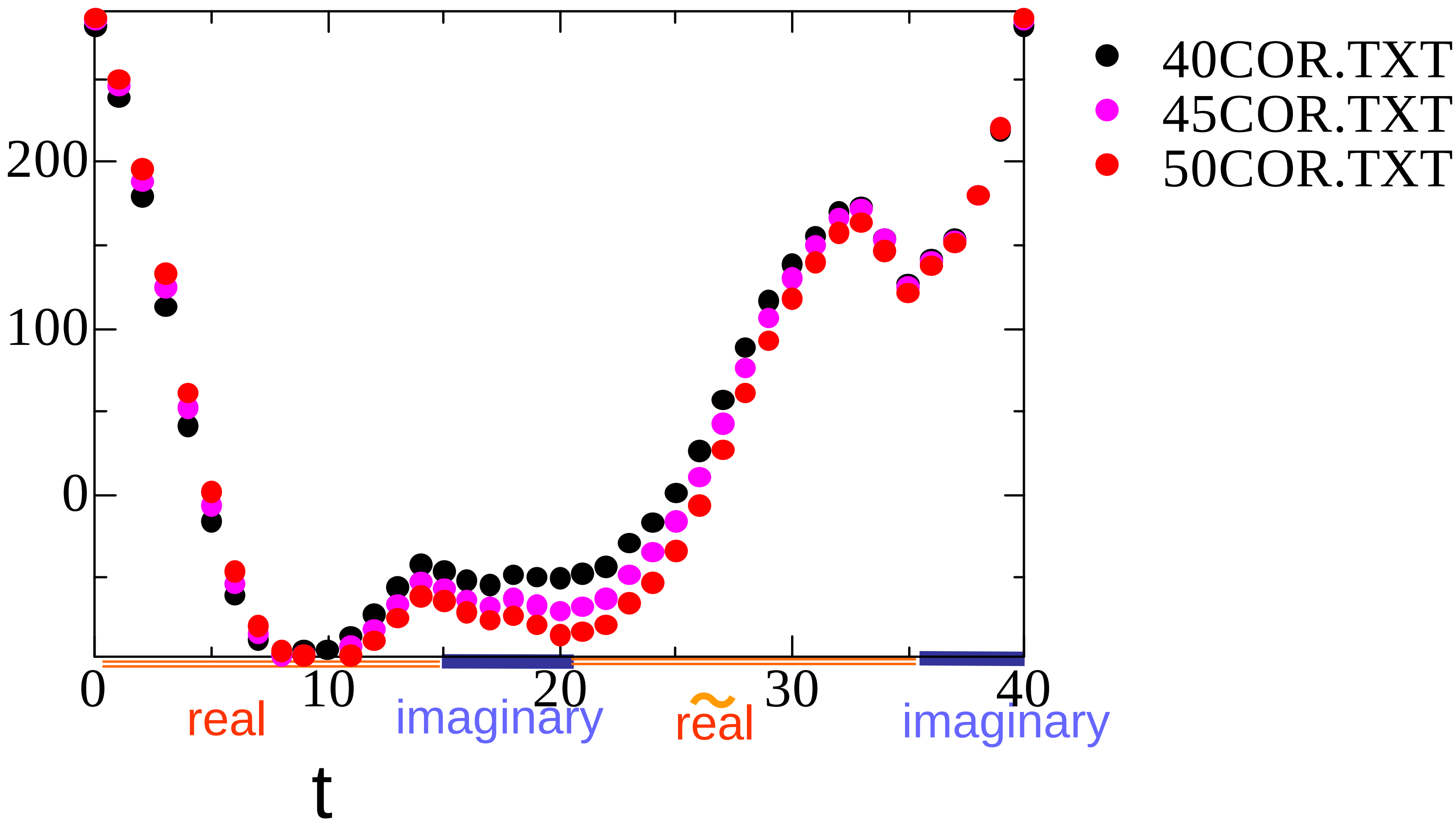
$$< \sum_{x} \phi^*(x, t=0)\phi(x,t) >$$



ma = 0.2, λ = 0.05

Average of 5000 steps

$$< \sum_{x} \phi^*(x, t=0)\phi(x,t) >$$

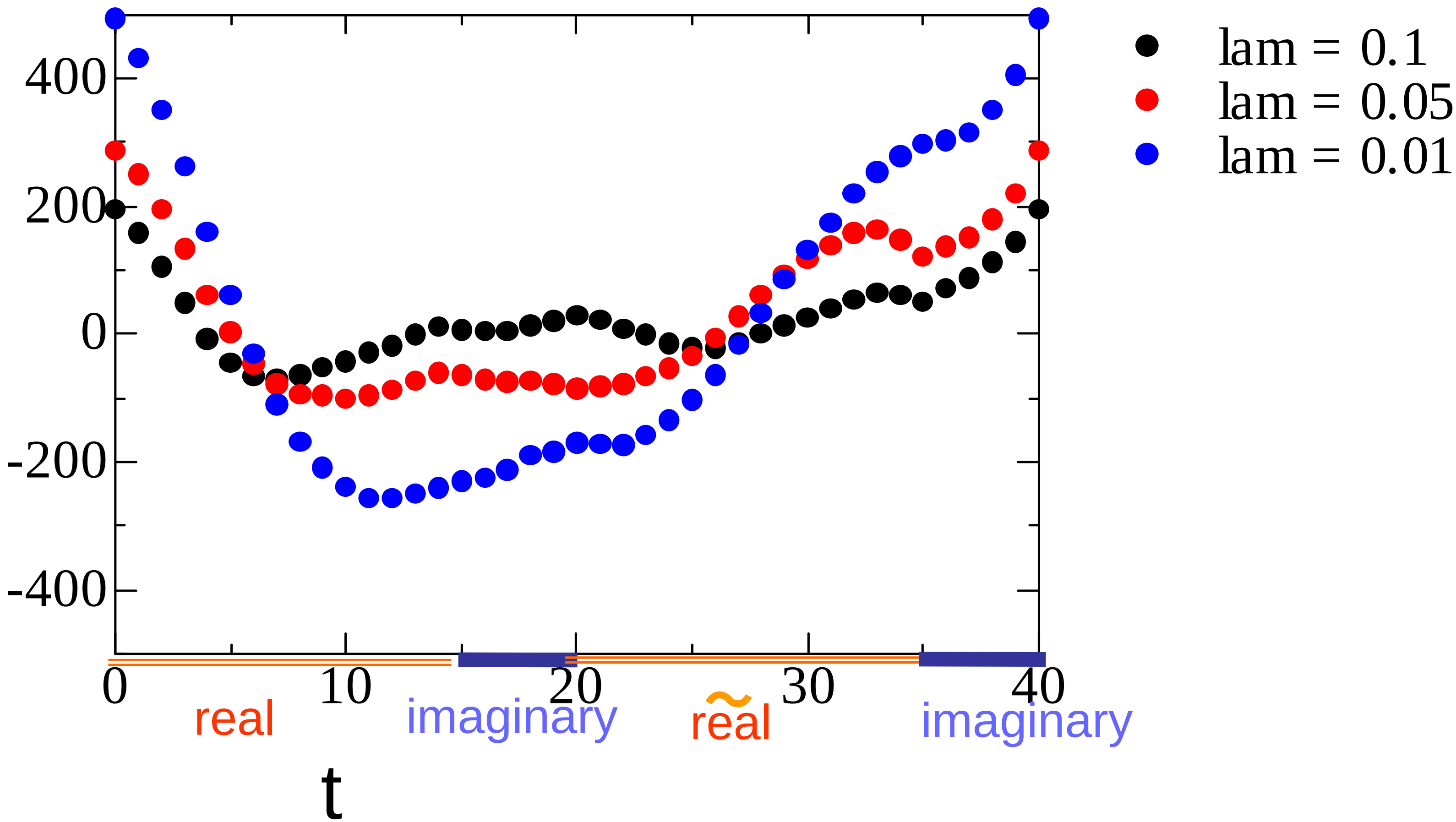


ma = 0.2, λ = 0.05

Average of 5000 steps

$$< \sum_{x} \phi^*(x, t=0)\phi(x,t) >$$

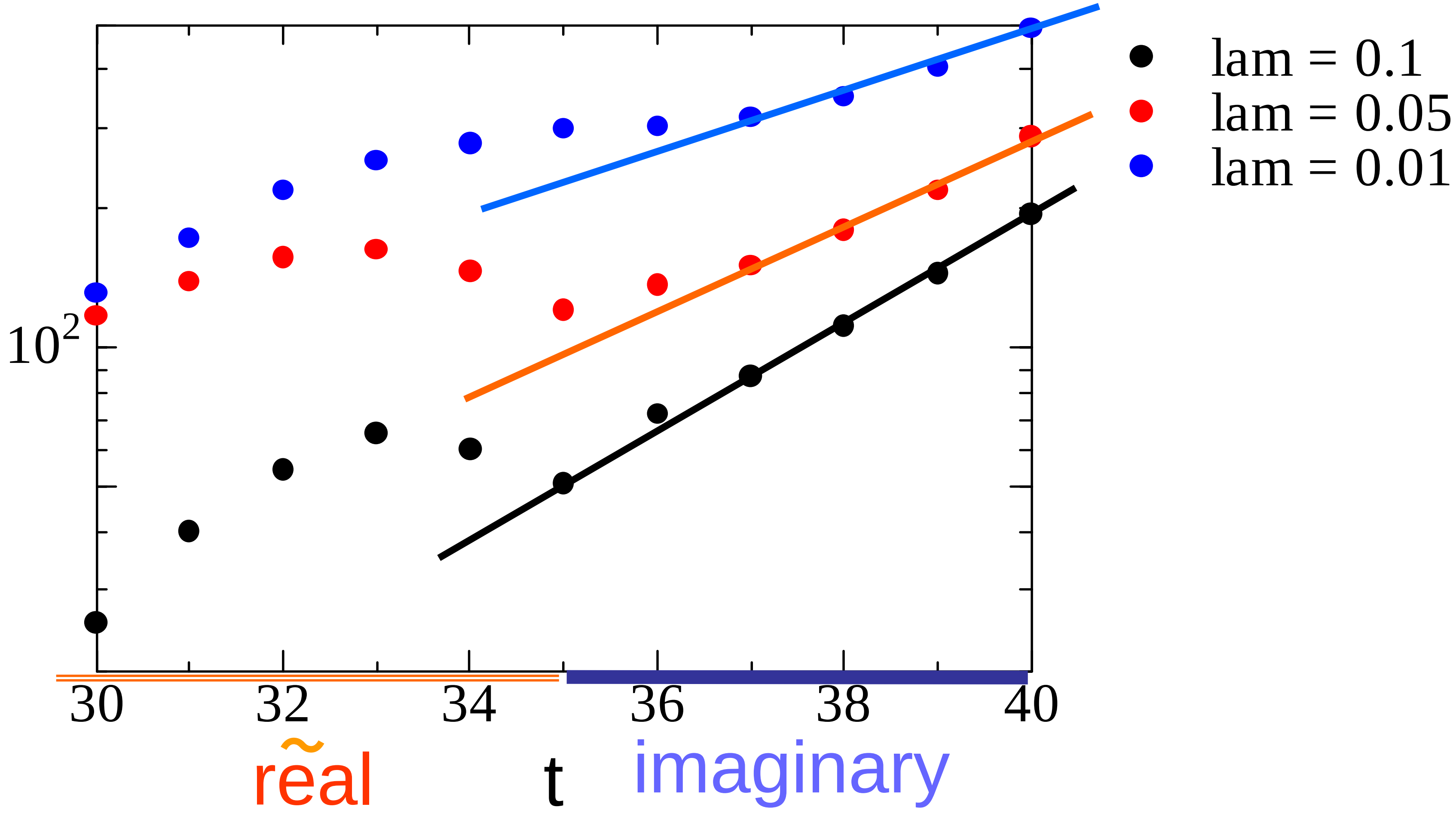
coupling λ dependences



ma = 0.2

50-th average of 5000 steps

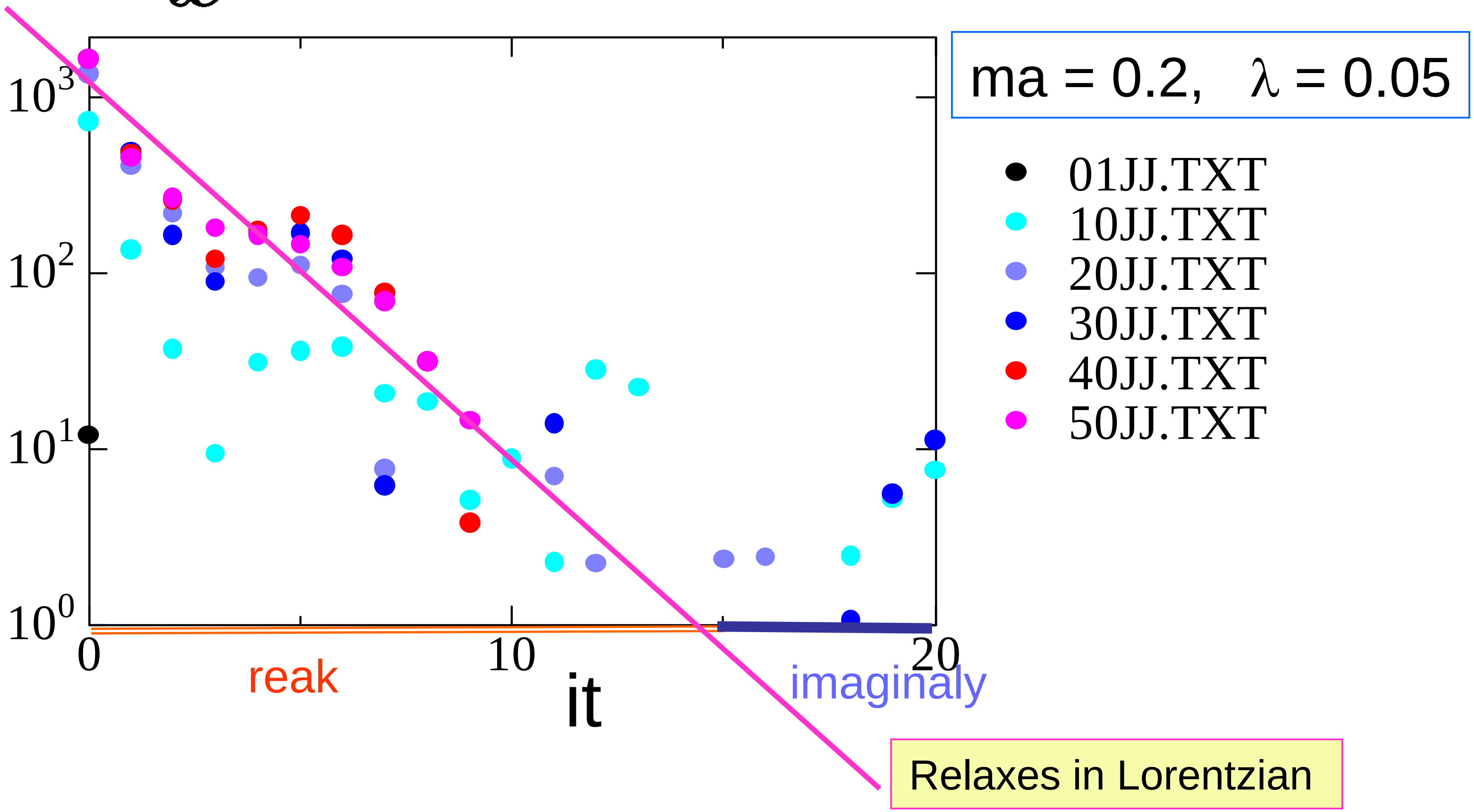
$$\log < \sum_{x} \phi^*(x, t=0)\phi(x,t) >$$



ma = 0.2

50-th average of 5000 steps

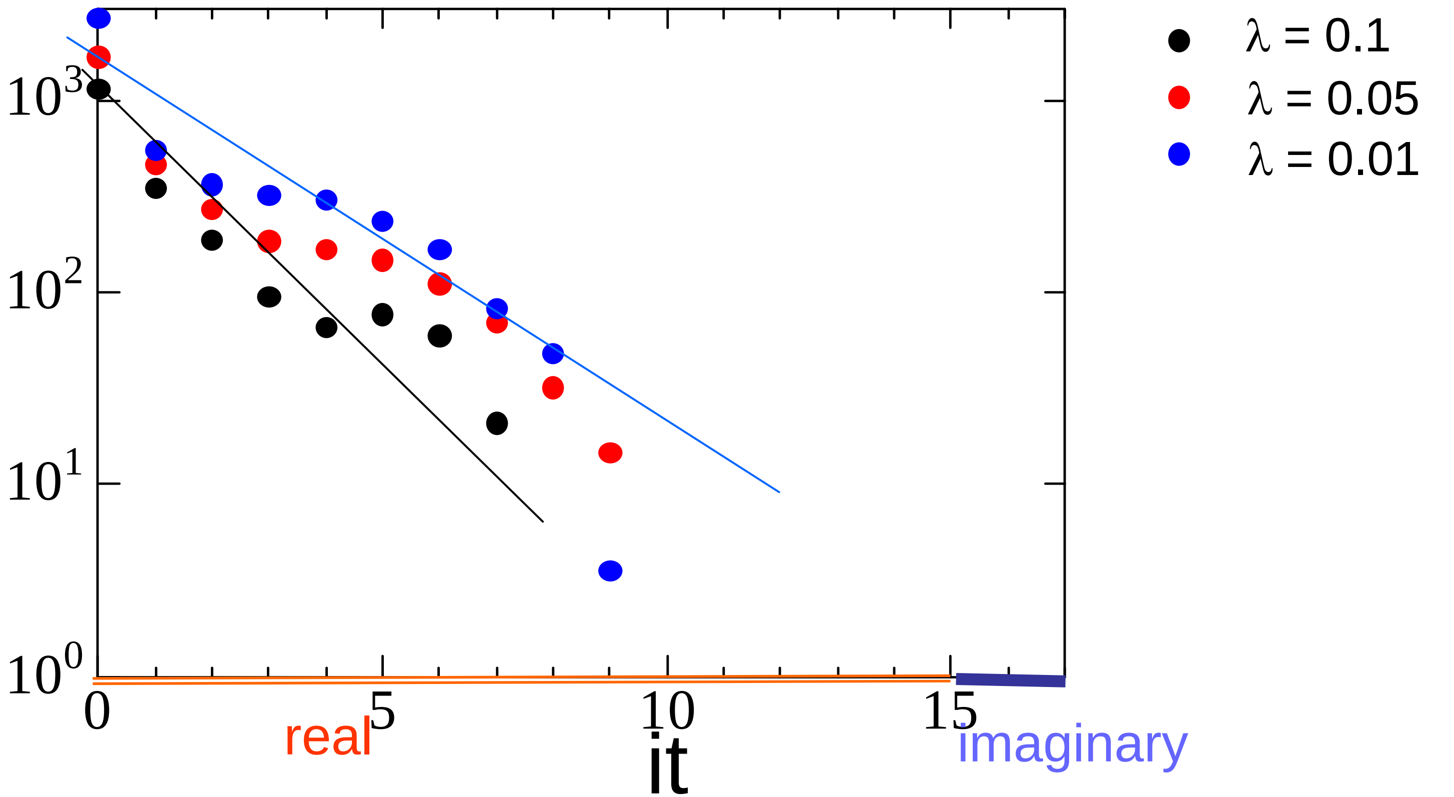
$$< \sum_{\boldsymbol{x}} J_x(\boldsymbol{x}, t = 0) J_x(\boldsymbol{x}, t) >$$



$$\boldsymbol{J} = \mathrm{Im}\left(\phi^*(\boldsymbol{x}, t) \nabla \phi(\boldsymbol{x}, t)\right)$$

Average of 5000 steps

$$< \sum_{\boldsymbol{x}} J_x(\boldsymbol{x}, t=0) J_x(\boldsymbol{x}, t) >$$



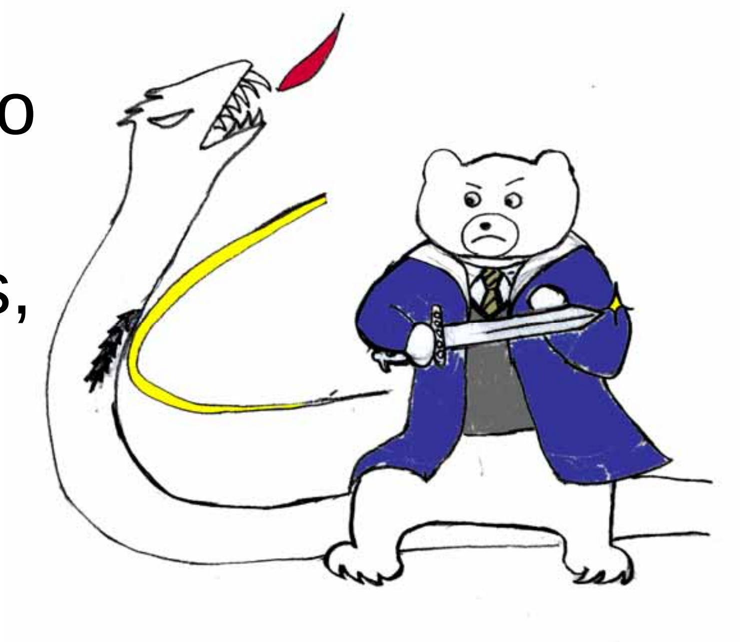
$$\boldsymbol{J} = \mathrm{Im}\left(\phi^*(\boldsymbol{x}, t)\nabla\phi(\boldsymbol{x}, t)\right)$$

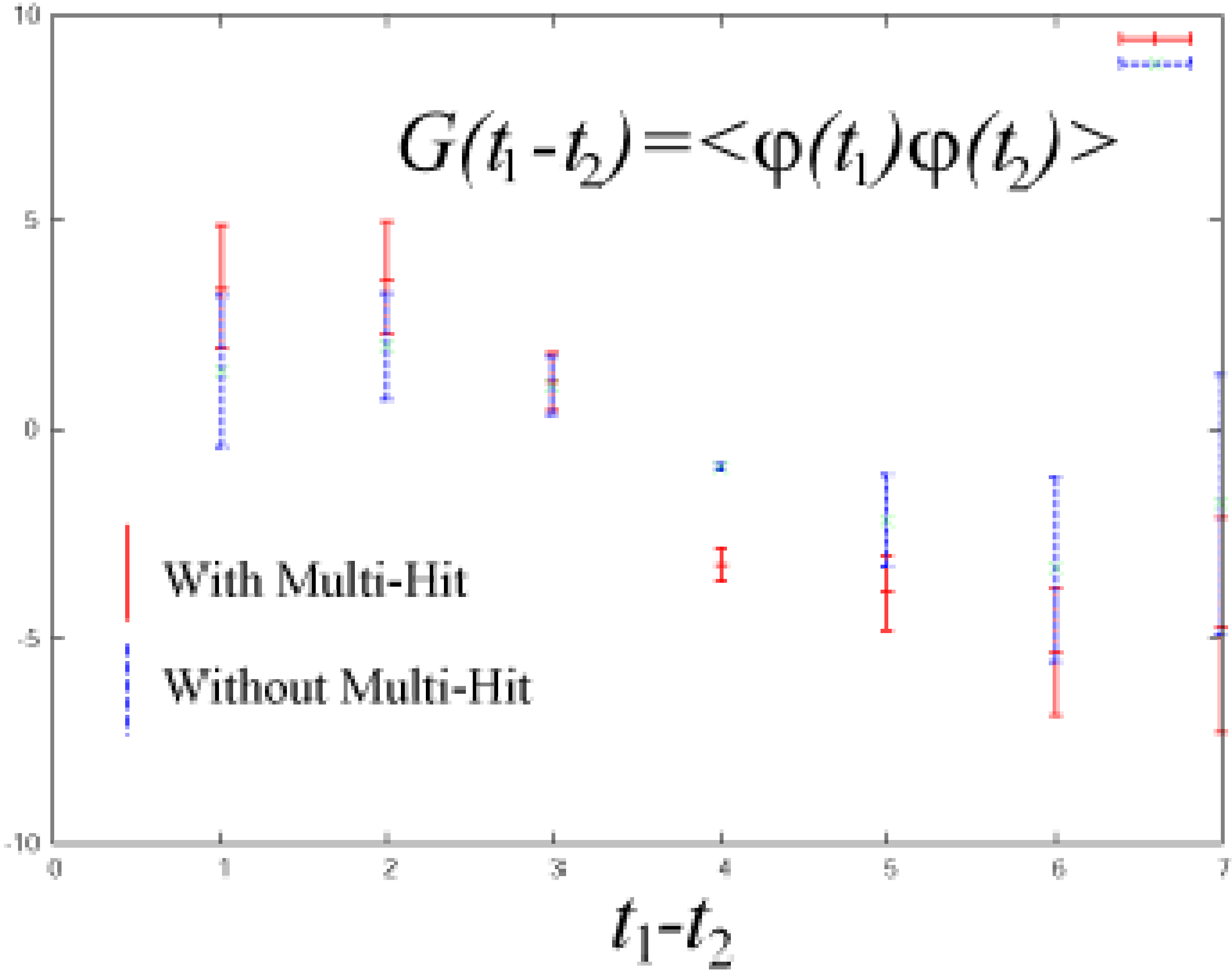
coupling λ dependences

50-th average of 5000 steps

Fighting against Errors

- Sign Problem: Suffer from +/- Cancelations
- We want to reduce pure noise parts.
- Luscher-Weise Multi-Hit Algorithm
  - employed by Meyer for Viscosity calculations
  - It can be considered as a version of Parisi-Petronzio -Rapuano
  - It does not work sometimes, i.e., parameter dependent




$G(t_1 - t_2) = <\varphi(t_1)\varphi(t_2)>$
10
5
0
-5
-10
0
1
2
3
4
5
6
7
With Multi-Hit
Without Multi-Hit
$t_1 - t_2$

Summary (Real Time Sim.)

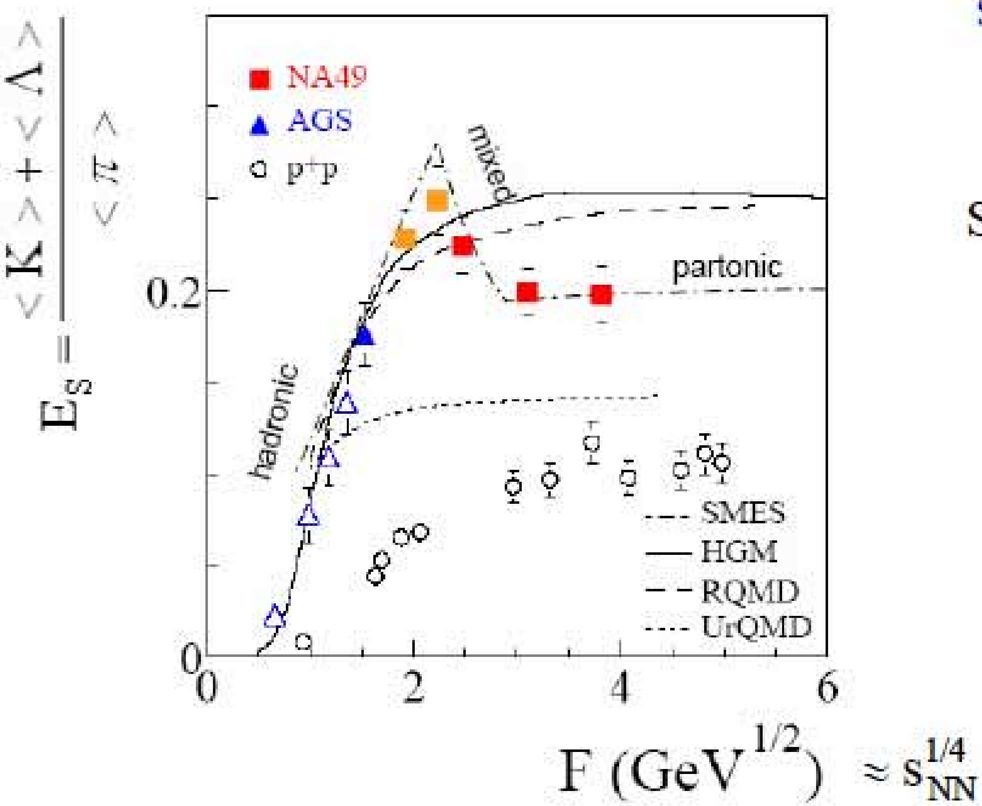
- Real Time Simulation for Equilibrium
- We need high Statistics and Improved Methods
- Possibility to calculate Transport-Coefficients

Sign Problem is Tough

- No Money, No Time, No Great Idea
- But I have a Dream that Sign Problem will be overcome someday.
- Let us Try any Possible Approaches (probably easier than Fermat's Last Theorem):

Backup Slides

Energy dependence – ratio of strange hadrons to pions



strangeness to pion ratio
peaks sharply at the SPS

SMES explanation:

- entropy, number of s,sbar quarks conserved from QGP to freezeout
- ratio of strange/nonstrange d.o.f. rises rapidly with T in hadron gas
- E_s drops to predicted constant level above the deconfinement threshold

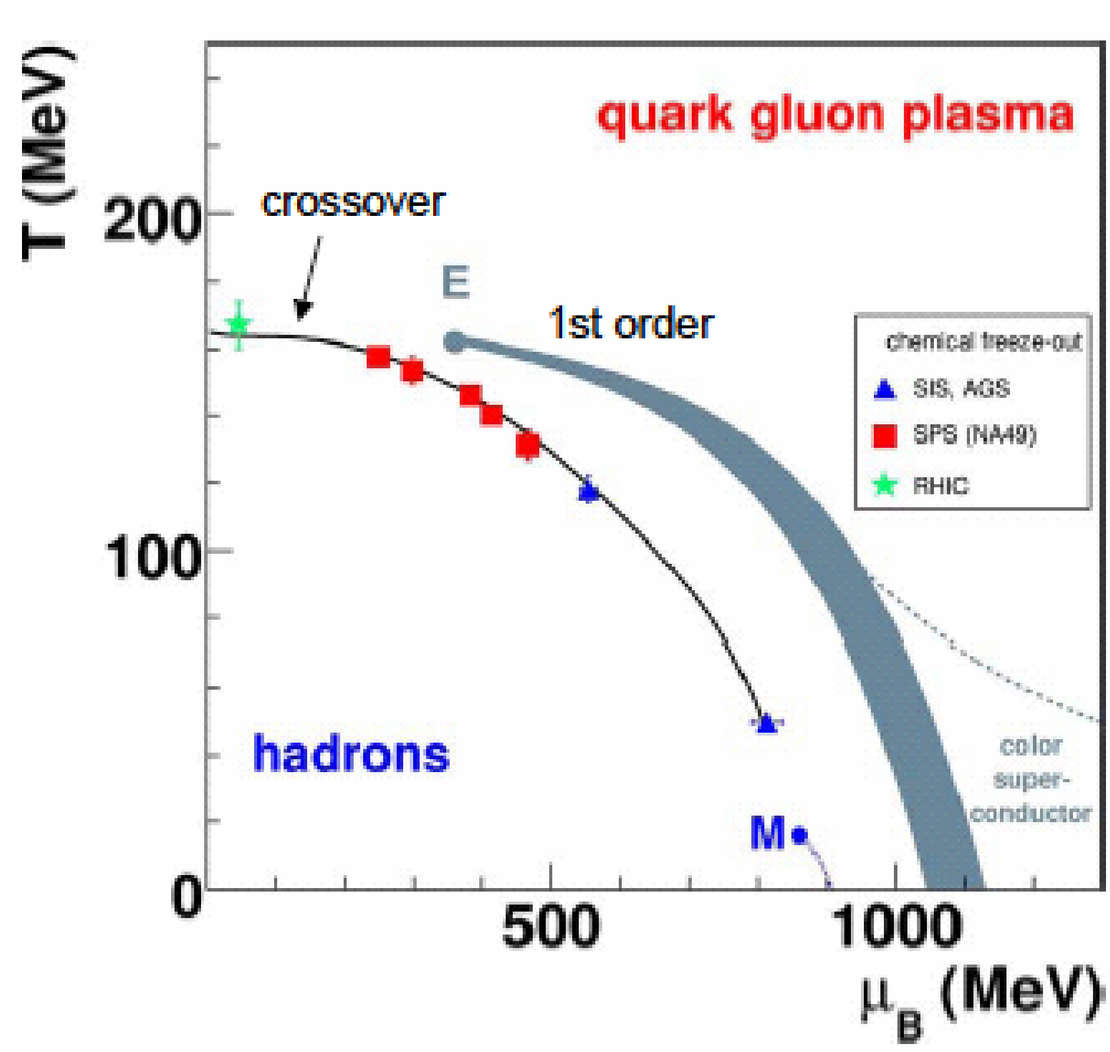
$$E_S \approx \frac{<N_S + N_{\bar{S}}>}{<\pi>} = \frac{0.74\, g_s}{g_u + g_d + g_g}$$

$$\approx \ 0.21$$

note: $<K> = 2\,(<K^+> + <K^->) = 4 <K_S^0>$

suggests onset of deconfinement at SPS

Recent CERN Experiments



Chemical Potential on the Lattice

In the continuum, μ appears as $p_4 \to p_4 - i\mu$

On the Lattice

$$2\sin\frac{pa}{2} \sim pa$$

Then

$$e^{\pm i p_4 a} \to e^{\pm i(p_4 - i\mu)a}$$

$$= e^{\pm i p_4 a} e^{\pm \mu a}$$

Calculation of $Tr\left(D_0^{-1}\Delta D\right)^l$

- (Gaussian) Noise Method
  - Almost equivalent to the Pseudo-Fermion Method

$$\overline{O} \equiv \frac{\int d\phi^* d\phi \; O e^{-S_\phi}}{\int d\phi^* d\phi \; e^{-S_\phi}}$$

$$S_\phi \equiv \left\langle \phi \,|\, D_0^\dagger D_0 \,|\, \phi \right\rangle = \left\langle \eta \,|\, \eta \right\rangle = \eta^\dagger \eta$$

$$\eta \equiv D_0 \phi$$

$$\Rightarrow \quad \overline{\phi_i \eta_j^*} = \left(D_0^{-1}\right)_{ij}$$

$$TrD_0^{-1}\Delta D = \left(D_0^{-1}\right)_{ij}\Delta D_{ji} = \overline{\phi_i \eta_j^* \Delta D_{ji}}$$

$$= \overline{\eta_j^* \Delta D_{ji} \phi_i} = \overline{\eta^\dagger \Delta D \phi}$$

In a similar way,

$$Tr\left(D_0^{-1}\Delta D\right)^2 = \overline{\left(\eta^\dagger \Delta D \phi\right)^2} - \left(TrD_0^{-1}\Delta D\right)^2$$

In General

$$\overline{\left(\eta^{\dagger}\Delta D\phi\right)^{l}} = c^{(l)}(m_{1,} m_{2,} \cdots, m_{l})$$

$$\times \sum_{m} Tr\left(\Delta D D_{0}^{-1}\right)^{m_1} \times Tr\left(\Delta D D_{0}^{-1}\right)^{m_2}$$

$$\times \cdots \times Tr\left(\Delta D D_{0}^{-1}\right)^{m_l}$$

I calculated these coefficients till 10-th just for fun.

In the Reweighting method, we try to maximize

$$\left| \det\Delta(0)\, e^{-(S_g(\beta)-S_g(\beta_0))} \frac{\det\Delta(\mu)}{\det\Delta(0)} \right|$$

i.e., minimize

$$\left(S_g(\beta)-S_g(\beta_0)\right)+\sum_{l=1}^{\infty}\frac{(-1)^l}{l}Tr\left(D_0^{-1}\Delta D\right)^l$$

If we employ the hopping parameter expansion together with this expansion,,

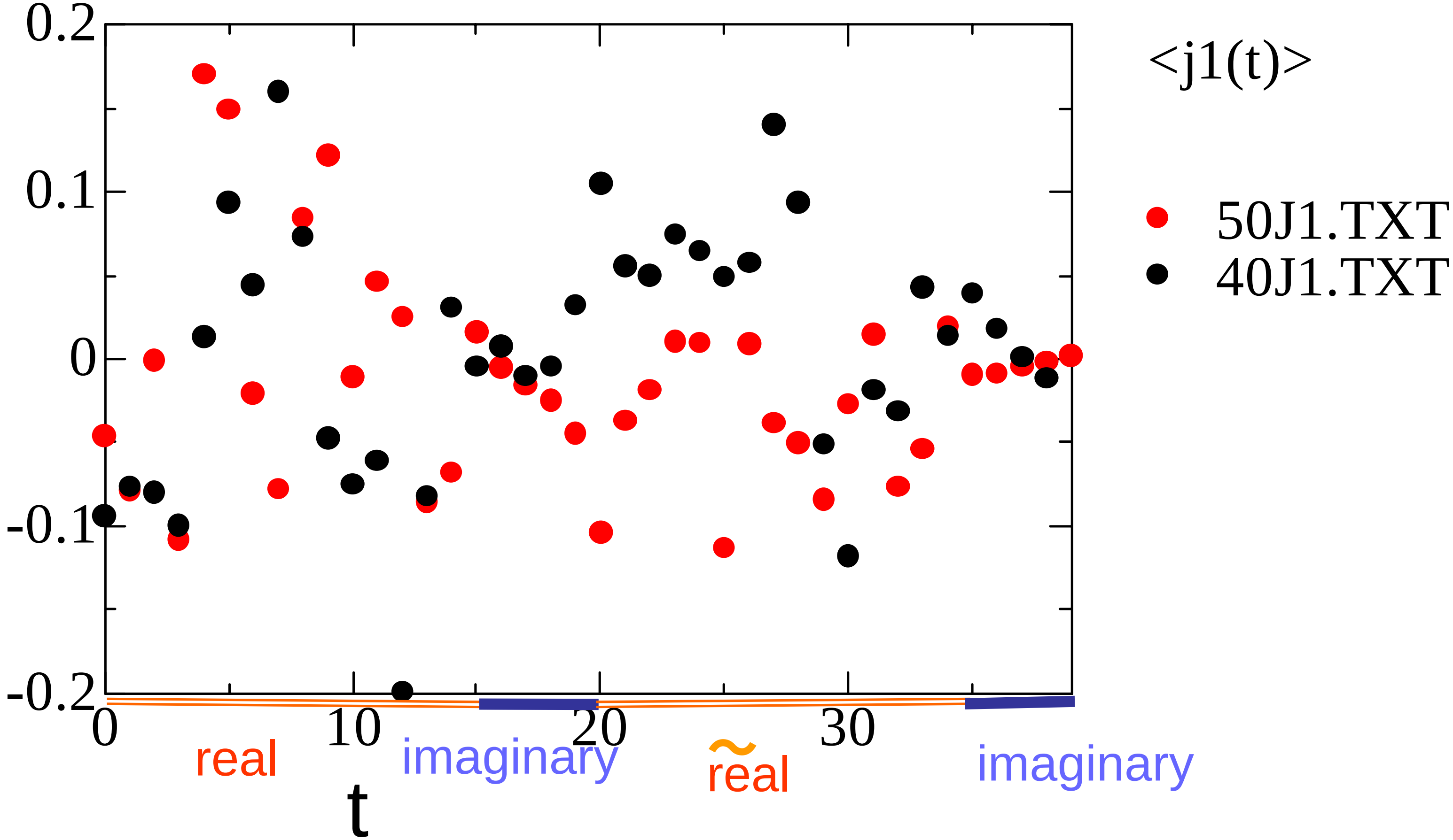
$$\Delta\beta\Box + C\Box_t + \cdots$$

which may help to find a good β_0. But we donot follow this direction.

$$< \sum_{x} J_x(x,t) >$$

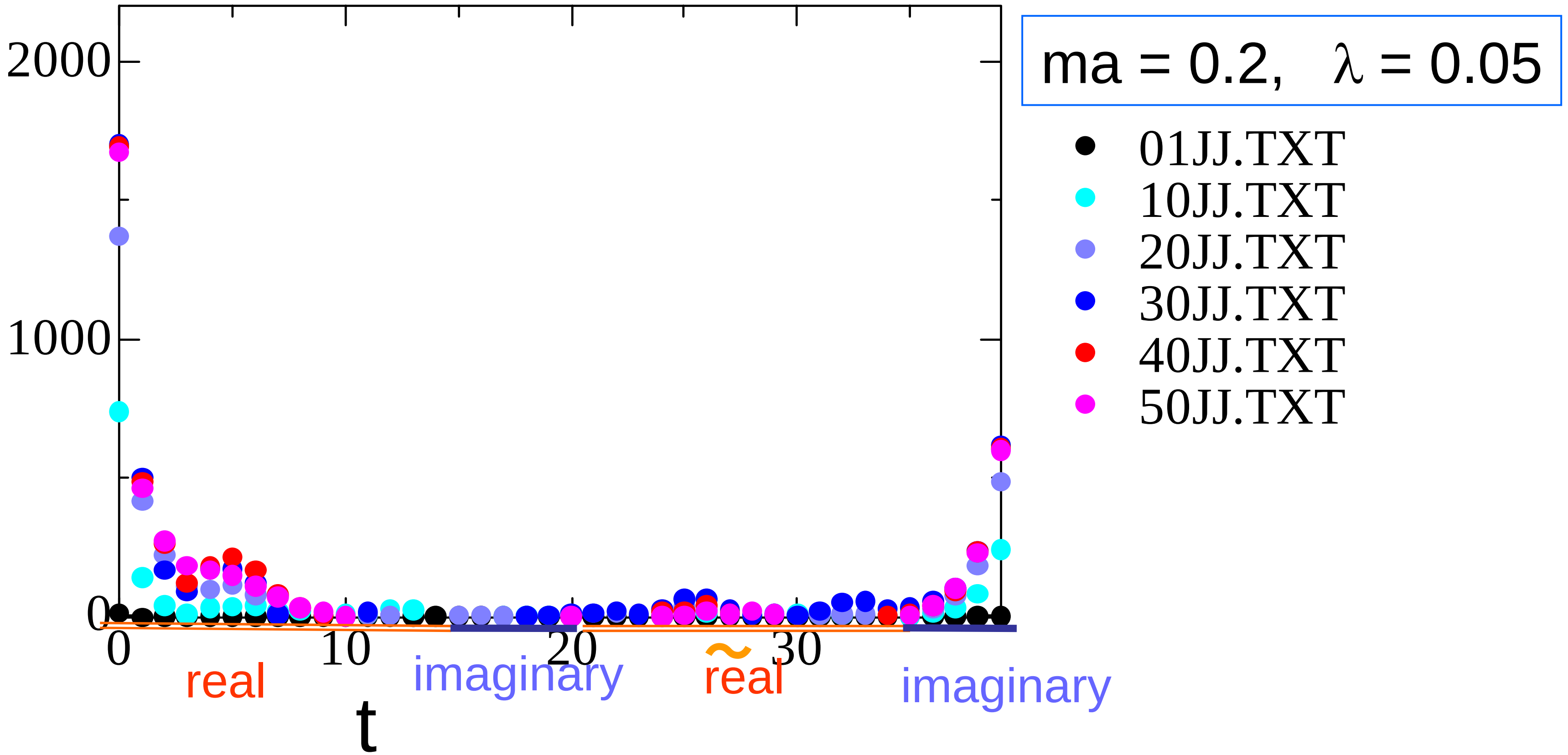
ma = 0.2, λ = 0.05

Average of 5000 steps.
(40th and 50th)



$$J = \mathrm{Im}\left(\phi^*(x,t)\nabla\phi(x,t)\right)$$

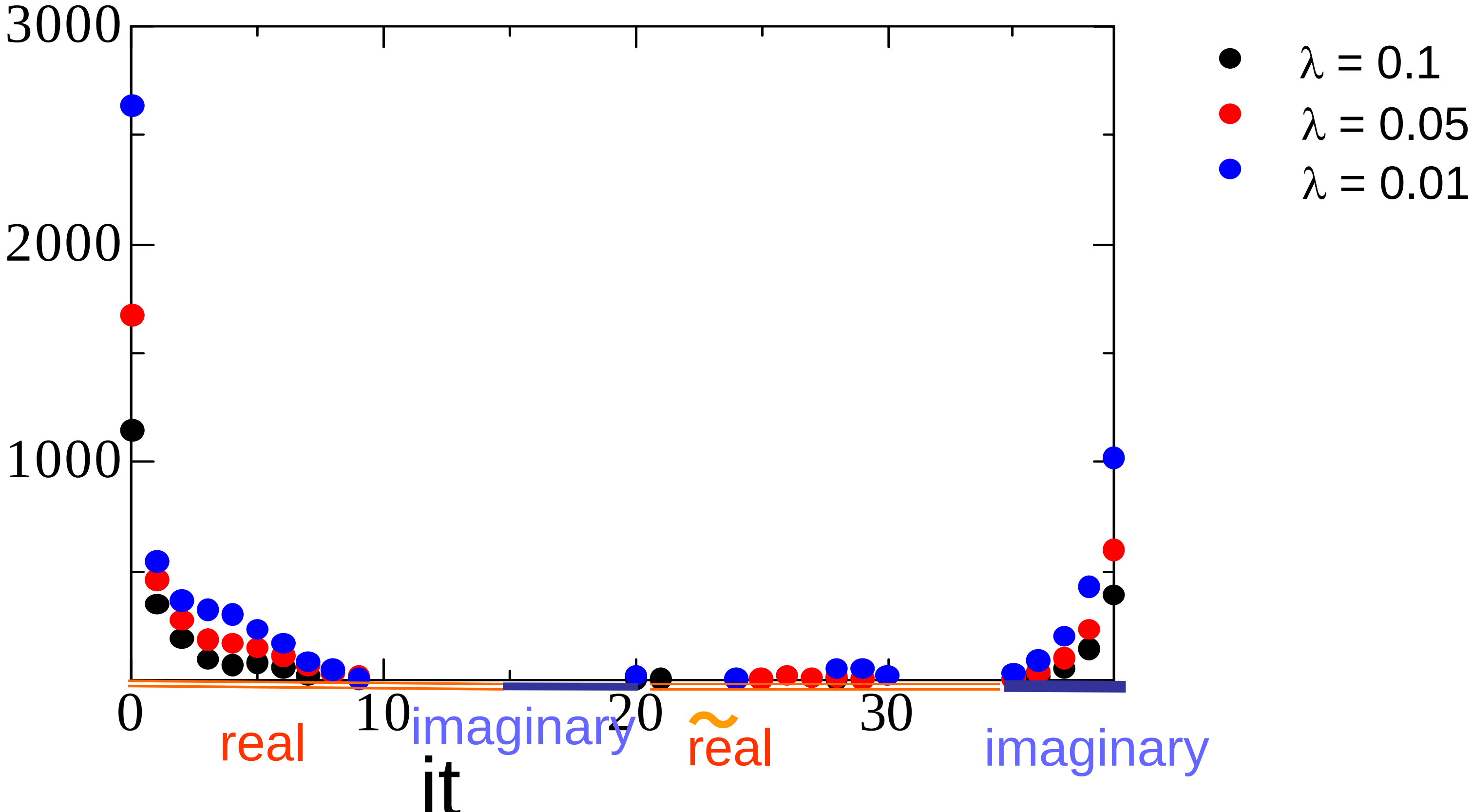
$$< \sum_{\boldsymbol{x}} J_x(\boldsymbol{x}, t=0) J_x(\boldsymbol{x}, t) >$$



$$\boldsymbol{J} = \mathrm{Im}\left(\phi^*(\boldsymbol{x}, t)\nabla\phi(\boldsymbol{x}, t)\right)$$

Average of 5000 steps

$$< \sum_{\boldsymbol{x}} J_x(\boldsymbol{x}, t=0) J_x(\boldsymbol{x}, t) >$$



$$\boldsymbol{J} = \mathrm{Im}\left(\phi^*(\boldsymbol{x}, t) \nabla \phi(\boldsymbol{x}, t)\right)$$

coupling λ dependences

We want to simulate numerically finite temperature system with real time.

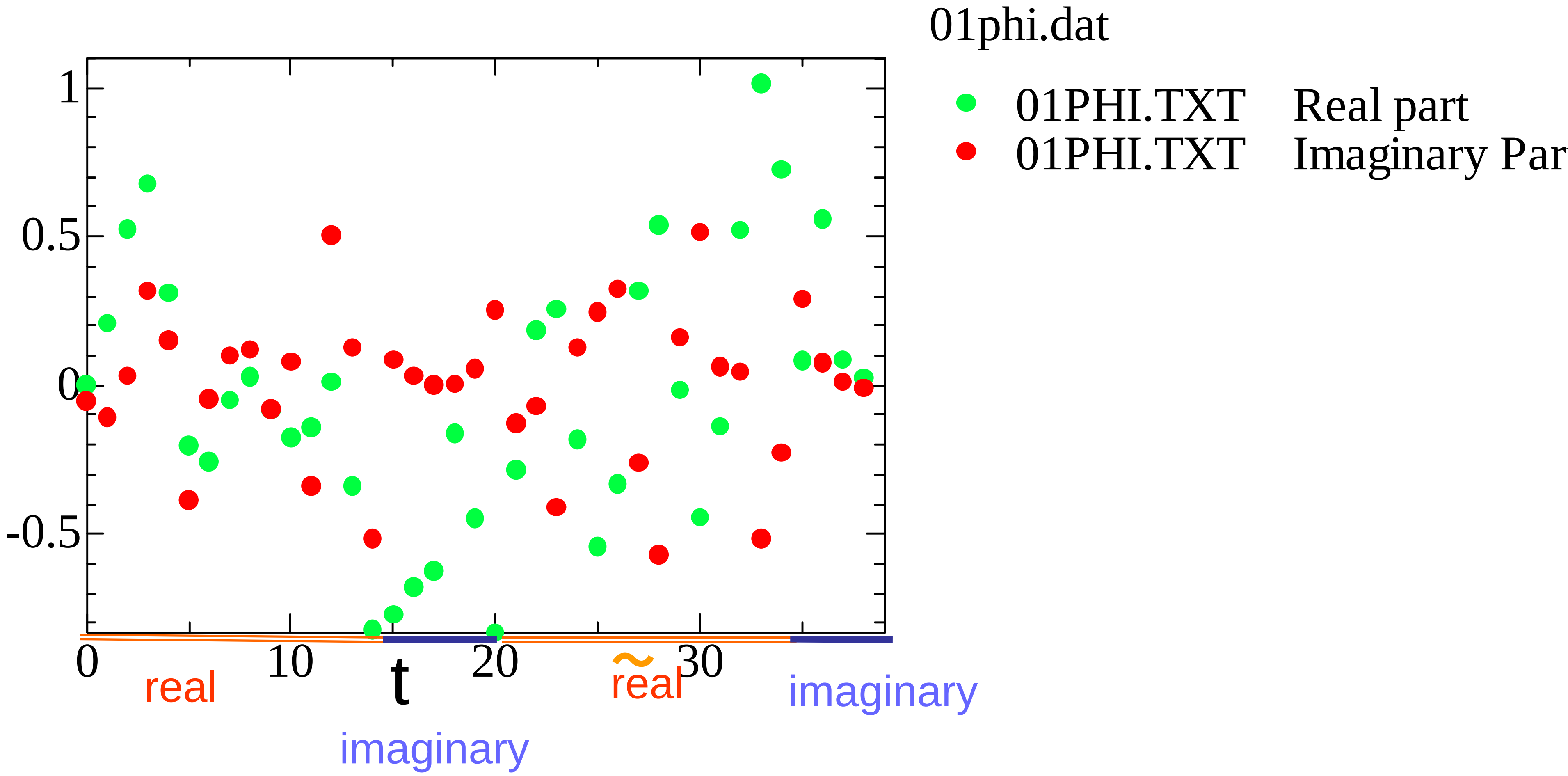
- Our results seem to converge even with Minkowski time.
- Current correlation ⇒ relaxation-like behavior appears

conductivity ?

- Coupling dependence
- Need to check
  - Contour dependence
  - Tilt dependence
  - Consistency with the results of imaginary time method

$$< \sum_{x} \phi(x,t) >$$

Hotstart first 5000 average



ma = 0.2, λ = 0.05