Sign Problems and Complex Actions

ECT* Trento

6 March 2009

IMAGINARY μ AND TAYLOR EXPANSION

Or

EXTRAPOLATION METHODS IN QCD
MIX & MATCH

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Plan

- TWO EXAMPLES
- QCD - (SOME) RESULTS FROM EXTRAPOLATION METHODS
  ANALITICITY PROPERTIES IN THE COMPLEX μ PLANE
- MIX & MATCH: POSSIBILITIES FOR
  - Radius of Covergence/Endpoint
  - Analysis of the hadronic phase around freezout

U(3) IN (0+1) DIMENSIONS

Fermionic determinant can be reduced to a $N_c \times N_c$ matrix

$$\det M = 2^{-nN_c} \det[2 + e^{n\mu} U + e^{-n\mu} U^\dagger],$$

where $U \in U(N_c)$.

n is the number of lattice points.

$$Z_{N_f}(\mu_c, \mu) = \int_{U(N_c)} dU \det M = 2.$$

No baryons, no μ dependence...

However, sign problem:

$$\langle \delta(\theta - \theta') \rangle_{N_f=0} = \frac{1}{2\pi} \frac{\sinh(2nN_c\mu)}{\cosh(2nN_c\mu) - \cos(\theta)}$$

Splittorf, Verbaarschot, MpL; K. Splittorff at this workshop

EXTRAPOLATION METHODS AND QCD IN 0+1 DIMENSIONS

$\mu = i\mu_I$

$n(i\mu_I) = 0$ holds true in analiticity domain;

Analytic continuation $n(\mu) = 0$.

Exact result $n(\mu) = 0$

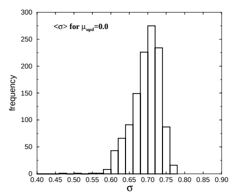
PERFECT!

GN in 3d at finite μ : NO Sign Problem

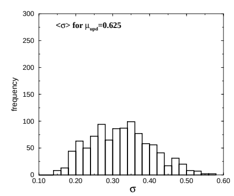
BUT : Overlap problem; failure of extrapolation methods from $\mu = 0$

The overlap problem as seen in the Gross Neveu model

Barbour,Hands,Kogut,Morrison, MPL, 1999



Broken phase



Symm. Phase

Distributions of the $<\sigma>=<\bar{\psi}\psi>$ fields

Situation can be ameliorated if a better starting point were used.

MBS2003 Catania

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$U(N_c)$ (1+0 DIMENSIONS:)

Severe sign problem

Silver Blaze

Success of extrapolation methods

GROSS NEVEU MODEL (2 + 1 DIMENSIONS)

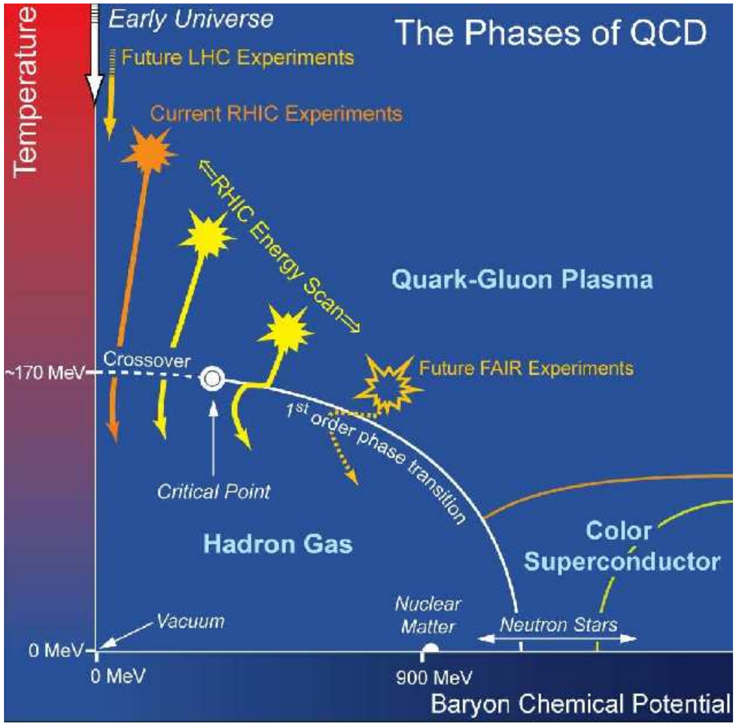
No sign problem

Dramatic overlap problem

Failure of extrapolation methods.

Difficult to make general statements!

THE PHASE DIAGRAM OF QCD



From NSAC Long Range Plan

Computational Schemes

$$\mathcal{Z} = \int d\phi d\bar{\psi} dU e^{-S(\phi,\bar{\psi},U)}; S(\phi,\bar{\psi},U) = \int_0^{1/T} dt \int d^d x \mathcal{L}(\phi,\bar{\psi}U)$$

$$\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \bar{\psi}(i\gamma_\mu D_\mu + m)\psi + \mu\bar{\psi}\gamma_0\psi$$

Two options:

1. Integrate out gluons first:

$$\mathcal{Z}(T,\mu,\bar{\psi},\psi,U) \simeq \mathcal{Z}(T,\mu,\bar{\psi},\psi) \longrightarrow$$

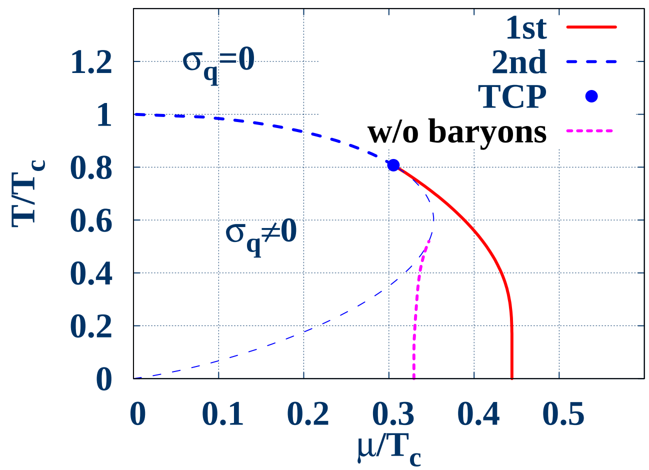
   effective approximate fermion models

2. Integrate out fermions exactly as S is bilinear in $\psi, \bar{\psi}$
   $S = S_{YM}(U) + \bar{\psi}M(U)\psi$

$$\mathcal{Z}(T,\mu,U) = \int dU e^{-(S_{YM}(U) - log(detM))} \longrightarrow$$

   starting point for numerical calculations

The phase diagram of QCD at strong coupling



Kawamoto et al, 2005

Importance Sampling and The Positivity Issue

$$\mathcal{Z}(T,\mu,U) = \int dU e^{-(S_{YM}(U) - log(\det M))}$$

$\det M > 0 \rightarrow$ Importance Sampling MonteCarlo Simulations

To assess sign problem consider $M^{\dagger}(\mu_B) = -M(-\mu_B)$

- $\mu = 0 \rightarrow \det M$ is real
  Particles-antiparticles symmetry : MC Simulations OK
- Imaginary $\mu \neq 0 \rightarrow \det M$ is real
  (Real) Particles-antiparticles symmetry : MC Simulations OK
- Real $\mu \neq 0$ Particles-antiparticles asymmetry
  $\rightarrow \det M$ is complex in QCD

QCD with a real baryon chemical potential: Extrapolate from the accessible region

$$Real \mu = 0, Im \mu \neq 0$$

QCD at nonzero Baryon Density: Extrapolation Methods

Multiparameter Reweighting ($\mu = 0$):

Fodor, Katz, Csikor, Egri, Szabo, Toth

Derivatives ($\mu = 0$):

Gupta, Gavai and collaborators; MILC; QCD-Taro

Expanded Reweighting ($\mu = 0$)

Bielefeld-Swansea

Analyitic continuation from Imaginary μ

Strong Coupling QCD *MpL*

Dim. Reduced QCD *Laine, Hart, Philipsen*

QCD *de Forcrand, Philipsen,Kratochvila*

D'Elia, MpL, Di Renzo

Azcoiti, Di Carlo, Galante, Laliena, Staggered

Luo et al. Wilson

Models *Giudice, Papa, D'Elia, Cosmai; de Forcrand, Kim....*

Analytic Continuation

- Analytically continue μ to complex values
- Perform simulation at imaginary μ
- In principle : if two analytic functions coincide within a subset of their analytic domain, they will coincide everywhere
- In practice : in a numerical calculations the functions will be evaluated on a finite ensemble of points and with a limited accuracy. We need a series representation and/or a phenomenological guess.

TOOLS FOR ANALYTIC CONTINUATION FROM IMAGINARY CHEMICAL POTENTIAL:

⋆ **Taylor Series**

$$O(\mu_I) = \sum_k a_k \mu_I^k \rightarrow O(\mu) = \sum_k (i)^k a_k \mu^k$$

MpL, Hart, Laine, Philipsen, de Forcrand and Philipsen, D'Elia and MpL, Azcoiti et al, Luo, Papa, D'Elia, Cosmai, Giudice, ..

⋆ **Fourier Analysis**

$$O(\mu_I) = \sum_k a_k exp(ki\mu_I) O(\mu) = \sum_k a_k exp(k\mu)$$

D'Elia, Di Renzo, MpL, de Forcrand and Kratovchila

⋆ **Pade' Approximants**

MpL; Cea, Cosmai, D'Elia, Papa, Gavai, Gupta

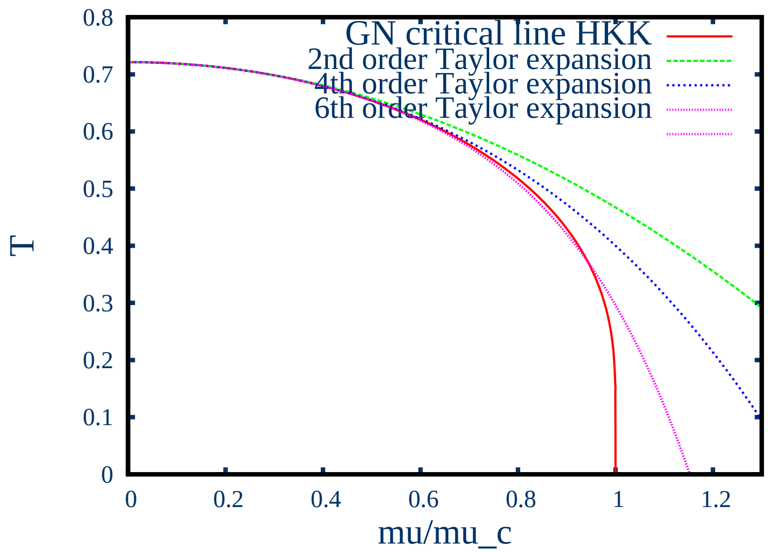
⋆ **Phenomenological Modeling**

Kaemfper, Bluhm

AN EXPLICIT CALCULATION:
The critical line of the 3d GN model

Hands, Kim, Kogut

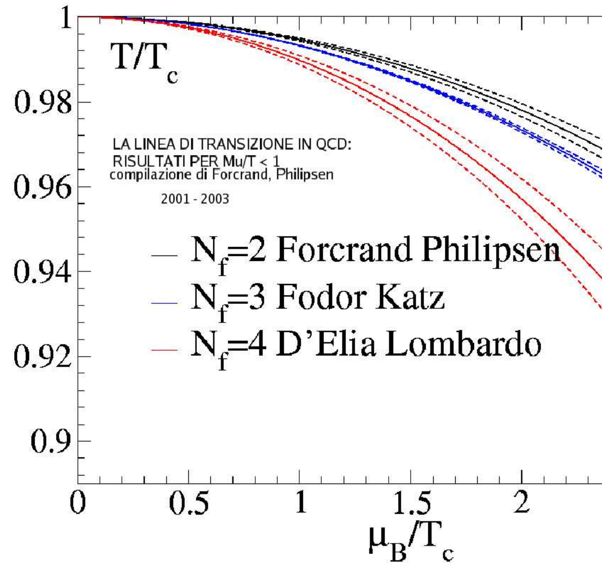
$$1-\mu/\Sigma_0 = 2T/\Sigma_0 ln(1+e^{-\mu/T})$$



$$\frac{d\mu_C(T)}{dT}_{T=0} = 0$$

THE (PSEUDO)CRITICAL LINE AT SMALL μ_B

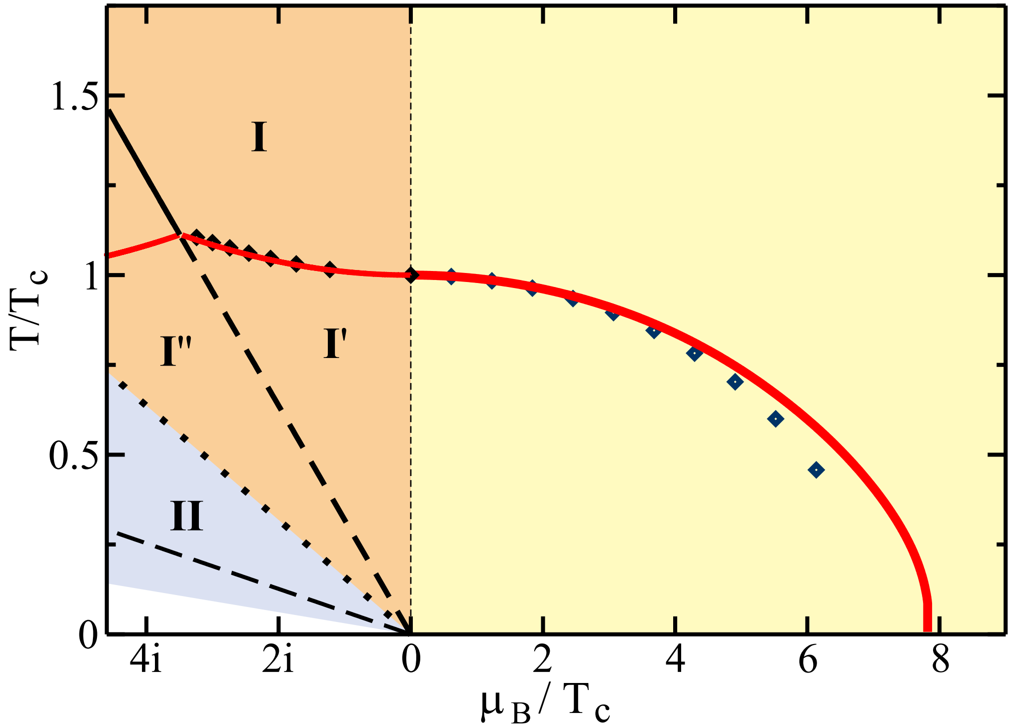
$$T = T_c(1 - K\mu_B^2/T_c^2)$$
$$K \propto N_f/N_c$$



Coefficient K in the Taylor expansion of the transition line, from $N_t = 4$
Compilation by Owe Philipsen, 2008

$$\frac{T_c(\mu)}{T_c(0)} = 1 - K(N_f, m_f)\left(\frac{\mu}{\pi T}\right)^2 + \mathcal{O}\left(\left(\frac{\mu}{\pi T}\right)^4\right) \quad .$$

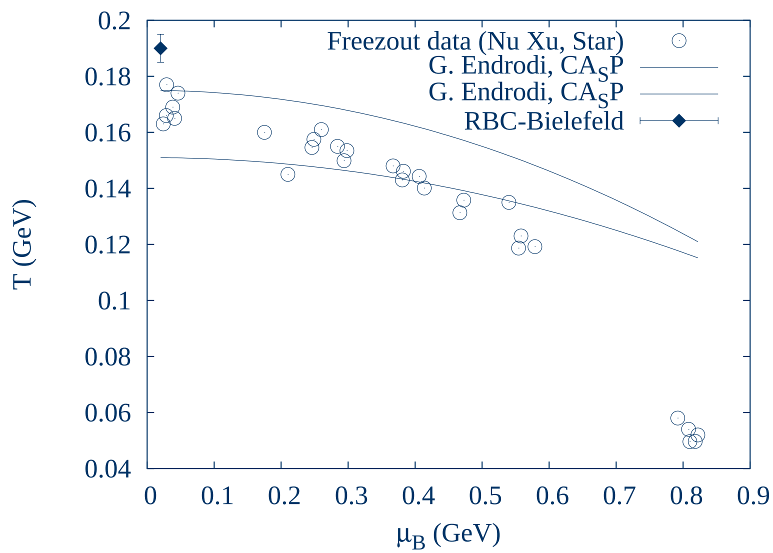
N_f	am	N_s	K	Action	β-Function	Method
2	0.1	16	0.69(35)	p4	non-pert.	Taylor+Rew.
	0.025	6,8	0.500(34)	stag.	2-loop pert.	Imag.
3	0.1	16	0.247(59)	p4	non-pert.	Taylor+Rew.
	0.026	8,12,16	0.667(6)	stag.	2-loop pert.	Imag.
	0.005	16	1.13(45)	p4	non-pert.	Taylor+Rew.
4	0.05	16	0.93(9)	stag.	2-loop pert.	Imag.
2+1	0.0092,0.25	6-12	0.284(9)	stag.	non-pert.	Rew.



Kämpfer, Bluhm 2008

4 flavor QCD Data from M. D'Elia, MpL 2004
Analytic continuation can be extended at lower T via Pade' (MpL 2005) or phenomenological models (Kampfer Bluhm 2008)

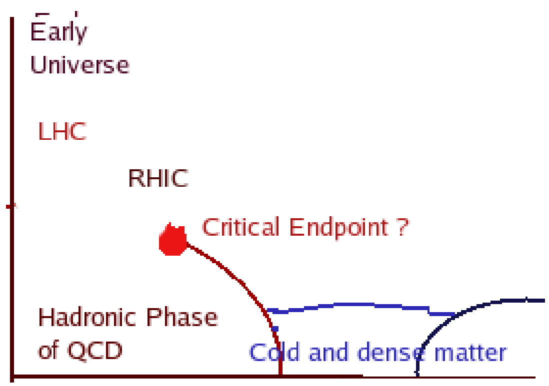
Status: Critical slope vs Freezout data



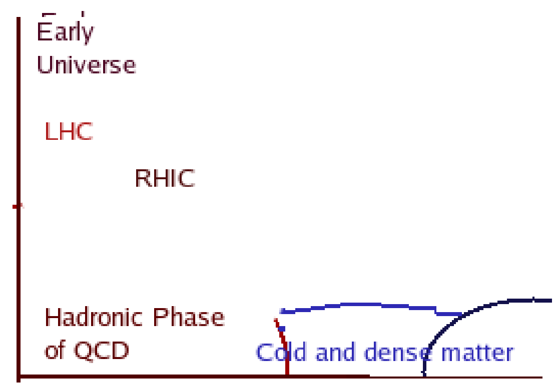
⋆ *Freezout data within the hadronic phase or the crossover region:OK*

⋆ *Accuracy for μ = 0 and small μ comparable*

THE CRITICAL ENDPOINT



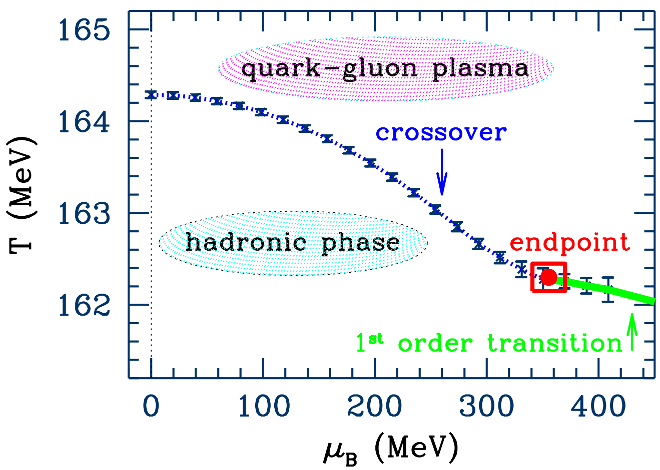
OR



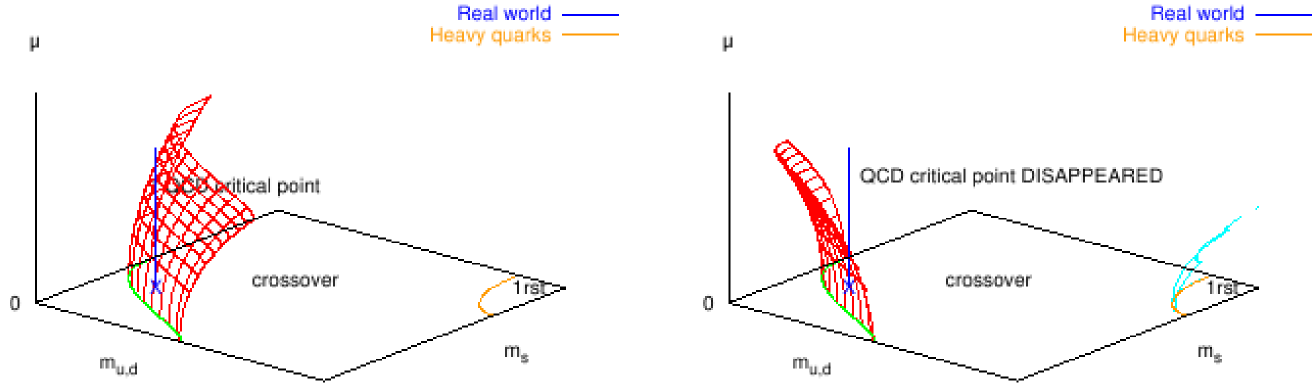
BOTH SCENARIO ARE COMPATIBLE

WITH MODEL CALCULATIONS AND UNIVERSALITY

Strategy 0 : Fodor Katz , reweighting from $\mu = 0$



STRATEGY I : FORCRAND-PHILIPSEN



Scenario I or Scenario II ? To decide, measure slope K in

$$\frac{m_c(\mu)}{m_c(0)} = 1 + K \left(\frac{\mu}{T}\right)^2 + \ldots$$

$K > 0$: Scenario I , critical endpoint at small μ_B
$K < 0$: Scenario II, NO critical endpoint at small μ_B

CURRENT RESULTS SUGGEST NO CRITICAL ENDPOINT FOR $\mu_B < 600 MeV$

STRATEGY II : GAVAI AND GUPTA, BIELEFELD-RBC

Series expansion for the pressure:

$$P(T,\mu_B) = P(T) + \frac{1}{2}\chi_B^{(2)}(T)\mu_B^2 + \frac{1}{4!}\chi_B^{(4)}(T)\mu_B^4 + \frac{1}{6!}\chi_B^{(6)}(T)\mu_B^6 + \frac{1}{8!}\chi_B^{(8)}(T)\mu_B^8 + \cdots,$$

The quark number susceptibility has the expansion

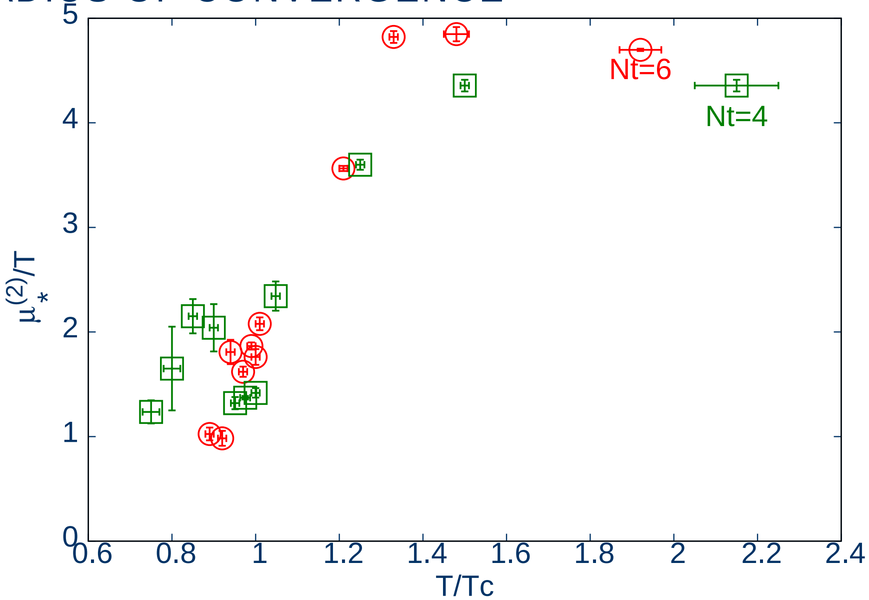
$$\chi_B(T,\mu_B) = \chi_B^{(2)}(T) + \frac{1}{2}\chi_B^{(4)}(T)\mu_B^2 + \frac{1}{4!}\chi_B^{(6)}(T)\mu_B^4 + \frac{1}{6!}\chi_B^{(8)}(T)\mu_B^6 + \cdots.$$

THIS SERIES IS EXPECTED TO DIVERGE AT THE QCD CRITICAL END POINT. RADIUS OF CONVERGENCE IS

$$\lim_{n\to\infty}\mu_*^{(n)} = \sqrt{\frac{1}{n(n-1)}\frac{\chi_B^{(n+2)}}{\chi_B^{(n)}}}.$$

The endpoint is the first singularity in the complex μ plane occurring at real μ. Coefficients should be all positive at large n

The radius of convergence



$$\frac{T^E}{T_c} = 0.94 \pm 0.01, \text{and} \frac{\mu_B^E}{T^E} = 1.8 \pm 0.1.$$

Extrapolation of this result to the thermodynamic limit, $L \to \infty$ on the coarse lattice :

$$\frac{T^E}{T_c} = 0.94 \pm 0.01, \text{and} \frac{\mu_B^E}{T^E} = 1.1 \pm 0.1.$$

sQGP, thermodynamics and complex μ

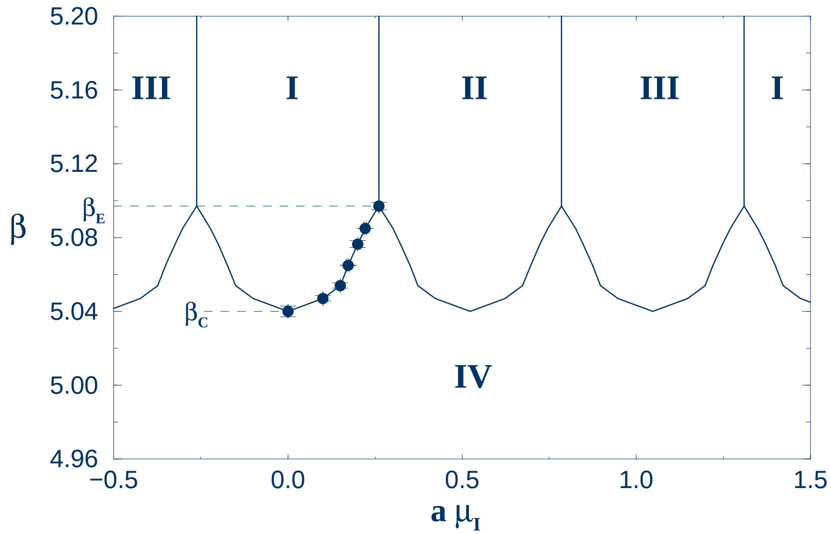
THE PHASE DIAGRAM IN THE IMAGINARY μ – T SPACE

$$Z(\mu_I/T) \equiv Z(V,T,i\mu_I/T) = \mathrm{Tr}\left(e^{i\mu_I N/T} e^{-\frac{H_{\mathrm{QCD}}}{T}}\right)$$

- N is a number operator: $Z(\mu_I/T)$ periodic in μ_I with period $2T\pi$; moreover a period $2T\pi/3$ is expected in the confined phase, where only physical states with N multiple of 3 are present.
- Observation (Roberge and Weiss) : $Z(\mu_I)$ is always periodic $2T\pi/3$, for any physical temperature!
- Low T : smooth periodicity
- High T : non-analytic behaviour with discontinuities at

$$\theta = 2\pi/3(k+1/2)$$

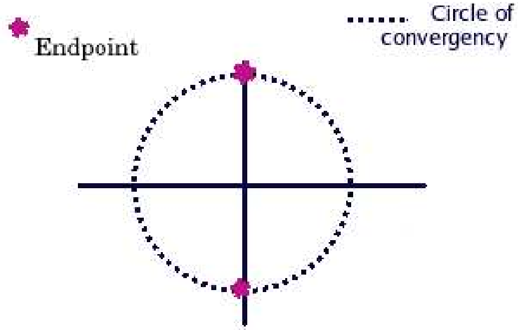
  corresponding to phase transitions from one Z_3 sector to the other.
- $P(\vec{x})e^{i\mu_I/T}$, instead of $P(\vec{x})$: μ_I/T fixes the preferred vacuum.



SKETCH OF THE PHASE DIAGRAM IN THE μ_I–β PLANE, $\mu_R = 0$

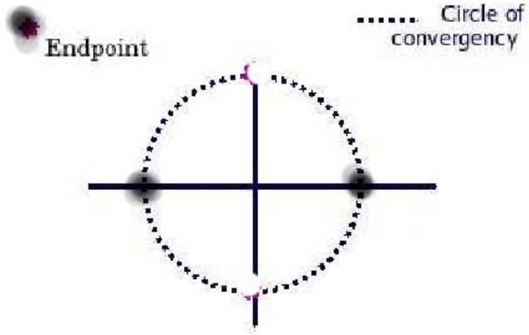
M. D'Elia, MpL

SINGULARITIES FOR COMPLEX μ, FIXED T



Endpoint of the RW Transition

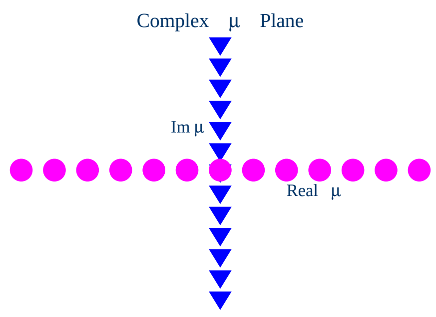
$$T_R > T_c$$



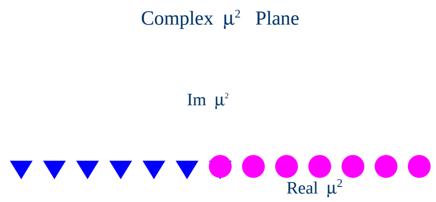
Endpoint of the Chiral Transition

$$T_\chi < T_c$$

BECAUSE OF THE QCD SYMMETRIES, THE COMPLEX μ_B PLANE

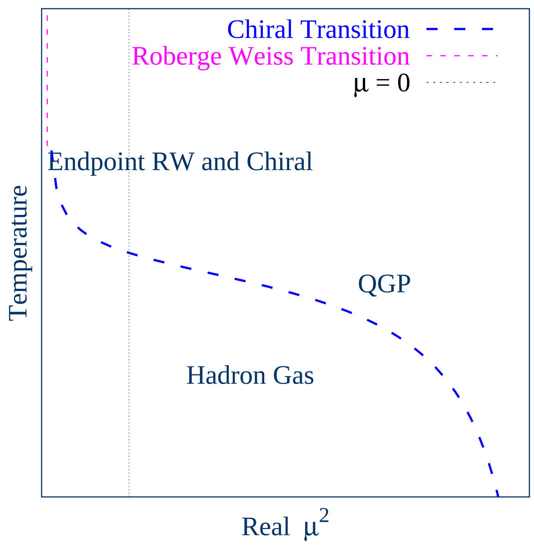


CAN BE MAPPED ONTO THE COMPLEX μ_B^2 PLANE



THERMODYNAMICS AND CRITICAL LINES in THE T μ^2 PLANE

Three regimes for thermodynamics:



- **Low Temperature**,

  away from critical lines:

  Hadron Gas

  $n(T, \mu) = K(T)\sinh(N_c\mu/T)$

- **In the sQGP region**:

  $p(T, \mu)$

  $= b(T)|t + a(T)(\mu^2 - \mu_c^2)|^{(2-\alpha)}$

  implying

  $n(T, \mu) =$

  $A(T)\mu(\mu^{c2} - \mu^2)^{(2-\alpha)}$

- **High Temperature**,

  away from critical line

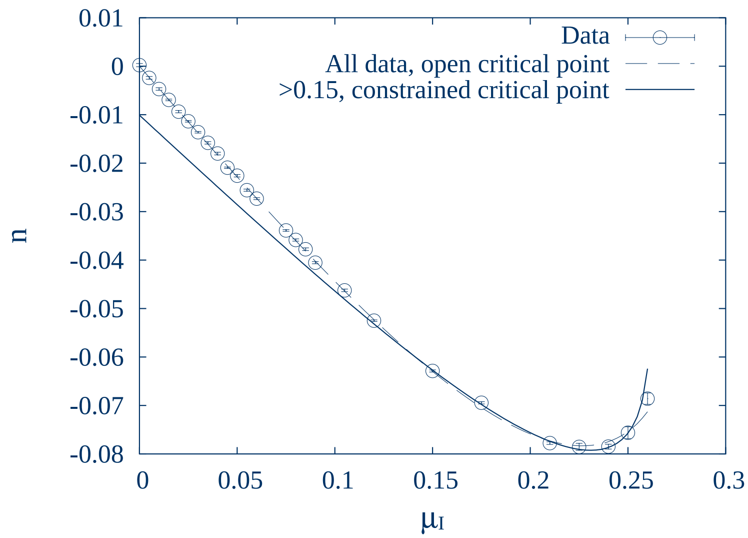
  Approach to Free Field

  $n(T, \mu) \rightarrow n_{SB}(T, \mu)$

CRITICAL BEHAVIOR AND THERMODYNAMICS AT THE ENDPOINT OF THE RW TRANSITION

Critical behavior at **imaginary** μ

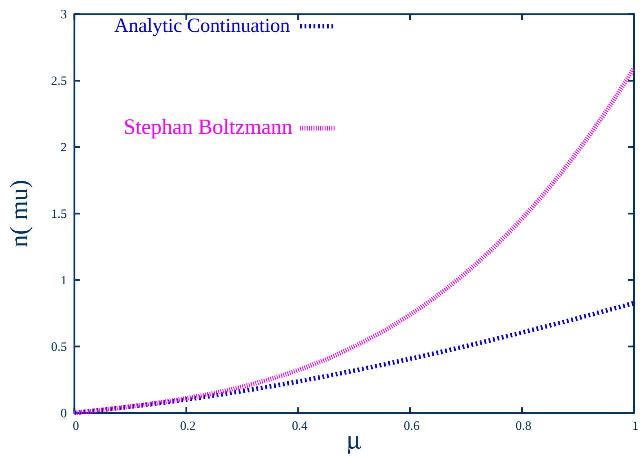
$$n(\mu_I) = A(T)\mu_I(\mu^{c2} - \mu_I^2)^{(2-\alpha)}$$



Continued to **real** μ..

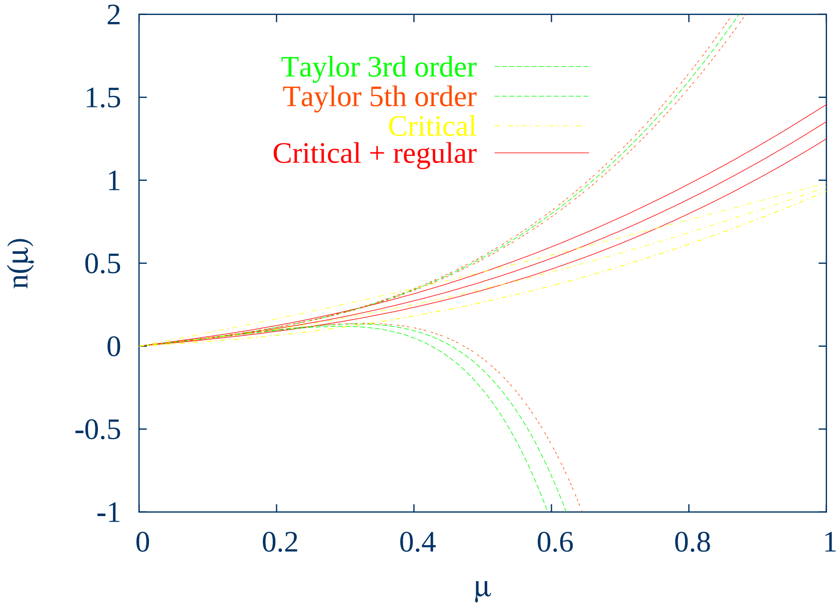
$$n(\mu) = A(T)\mu(\mu^{c2} + \mu^2)^{(2-\alpha)}$$

$$n_{SB}(\mu) = A\mu + B\mu^3) \rightarrow \alpha = 1$$



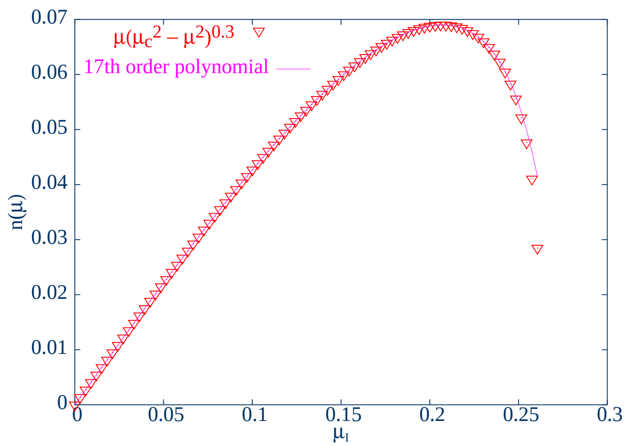
D'Elia, Di Renzo, Lombardo, 2007, QM2008

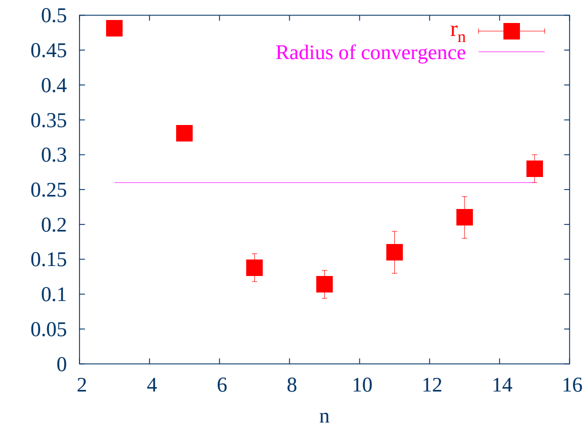
FINITE RADIUS OF CONVERGENCE OF THE TAYLOR EXPANSION = μ_{RW}



Taylor expanding the numerical result
at imaginary μ
M. D'Elia, F. Di Renzo, MpL 2007

And computing the radius of convergence





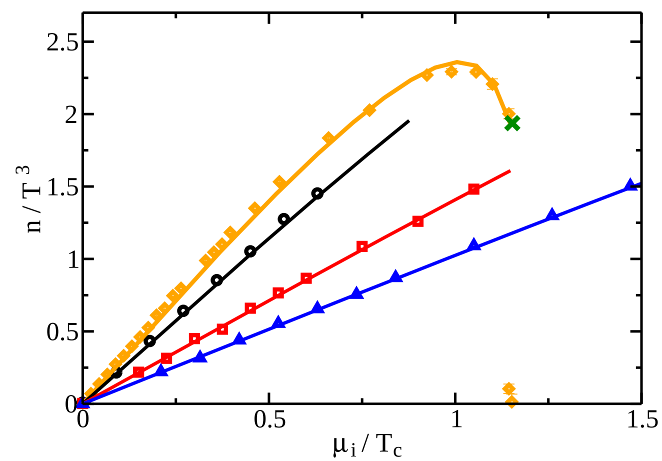
CAVEAT : the correct result might need many orders

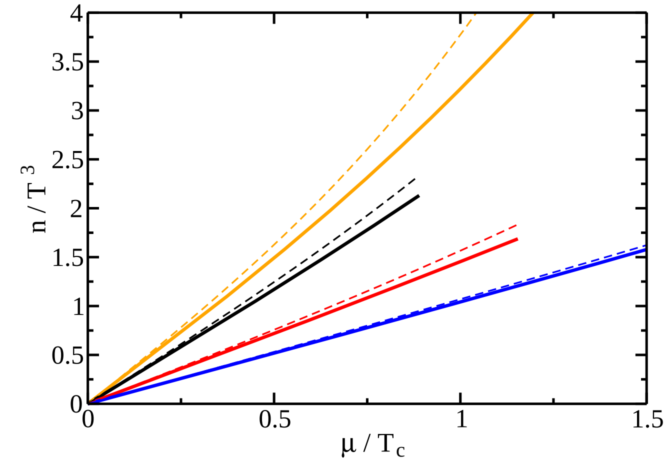
CRITICAL BEHAVIOR, THERMODYNAMICS, QUASIPARTICLE MODELS

Kämpfer, Bluhm Proposal

amenable to an easy comparison with lattice data.

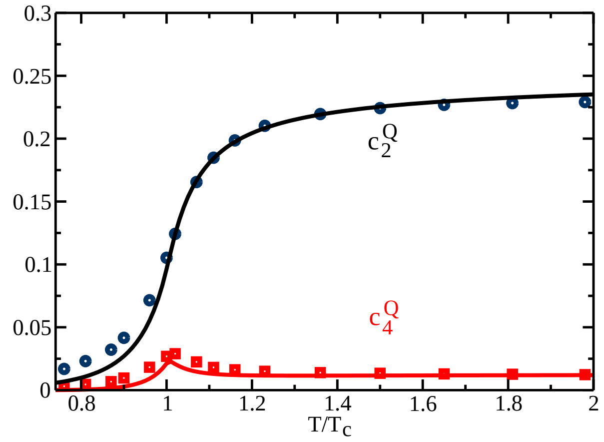
I. Quasiparticlemodel vs Imaginary Chemical Potential Lattice Data,
and analytic continuation to Real Chemical Potential

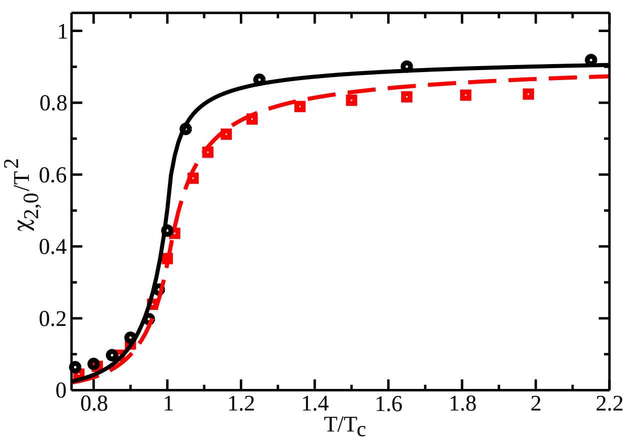




Kämpfer, Bluhm , 2007, QM2008

Quasiparticlemodel vs Taylor Coefficients





Crucial ingredients:

- Explicit dependence of the self-energy parts on $\mu_i = \mu_{u,d}$ and T
- Implicit dependence via the effective coupling $G^2(T, \mu_u, \mu_d)$.

$$\omega_i^2 = k^2 + m_i^2 + \Pi_i, \quad \Pi_i = \frac{1}{3}\left(T^2 + \frac{\mu_i^2}{\pi^2}\right) G^2(T, \mu_u, \mu_d)\,.$$

sQGP, thermodynamics and complex μ

Summarizing:

Interplay between the critical line at Im μ and thermodynamics

Modified Stefan-Boltzmann in the sQGP region

Amenable to cross checks with phenomenolgy

μ dependence of the mass spectrum

$\star$ Laboratory for the study of the radius of convergence

MIX & MATCH: POSSIBILITIES FOR

- Radius of Covergence/Endpoint
- Analysis of the hadronic phase around freezout

E. Laermann MpL

Based on the 2 Flavor QCD results of the Bielefeld Swansea Collaboration

STRATEGY:

1. Use numerical results from Taylor series.
2. Consider possible parametrizations motivated the singularity structure in the complex plane
3. Validate/check with imaginary chemical potential results
4. Estimate convergence range R of truncated Taylor series
5. Possibility of thermodynamics for $\mu < R$
6. Radius of convergence ¿ R

..iterate..

THERMODYNAMICS OF TWO FLAVOR QCD

T/T_c	c_2	$c_4 \times 10$	$c_6 \times 10^2$	c_2^I	$c_4^I \times 10$	$c_6^I \times 10^2$
0.76	0.0243(19)	0.238(61)	-1.12(121)	0.0649(6)	0.098(5)	0.23(9)
0.81	0.0450(20)	0.377(64)	1.98(141)	0.0874(8)	0.140(6)	0.44(10)
0.87	0.0735(23)	0.506(68)	1.69(155)	0.1206(11)	0.216(8)	0.60(13)
0.90	0.1015(24)	0.765(72)	2.06(159)	0.1551(14)	0.302(12)	0.83(18)
0.96	0.2160(31)	1.491(135)	4.96(260)	0.2619(21)	0.564(23)	1.47(37)
1.00	0.3501(32)	2.133(121)	-5.00(359)	0.3822(26)	0.839(28)	0.26(49)
1.02	0.4228(33)	2.258(118)	-4.49(312)	0.4501(27)	0.909(28)	0.02(44)
1.07	0.5824(23)	1.417(62)	-5.73(158)	0.5972(21)	0.741(17)	-0.75(26)
1.11	0.6581(20)	0.951(39)	-1.65(62)	0.6662(18)	0.618(11)	-0.18(10)
1.16	0.7091(15)	0.763(24)	-0.31(26)	0.7156(14)	0.564(6)	-0.03(4)
1.23	0.7517(16)	0.667(23)	-0.44(23)	0.7573(13)	0.527(5)	-0.06(3)
1.36	0.7880(11)	0.572(12)	-0.09(11)	0.7906(9)	0.495(3)	-0.03(1)
1.50	0.8059(10)	0.539(10)	-0.17(7)	0.8076(7)	0.477(2)	-0.05(1)
1.65	0.8157(8)	0.499(7)	-0.13(8)	0.8169(7)	0.461(2)	-0.05(1)
1.81	0.8203(8)	0.497(7)	-0.11(6)	0.8218(6)	0.452(1)	-0.05(1)
1.98	0.8230(7)	0.473(6)	0.03(4)	0.8250(6)	0.441(1)	-0.03(1)

C. R. Allton *et al.*, Bielefeld–Swansea, Phys. Rev. D **71** (2005) 054508

Partial sums of the Taylor series

$$n_q^K(T,\mu_q)/T^3 = \sum_1^K c_k(\mu_q/T)^{2k-1}$$

available data : K = 1, 2, 3.

Then

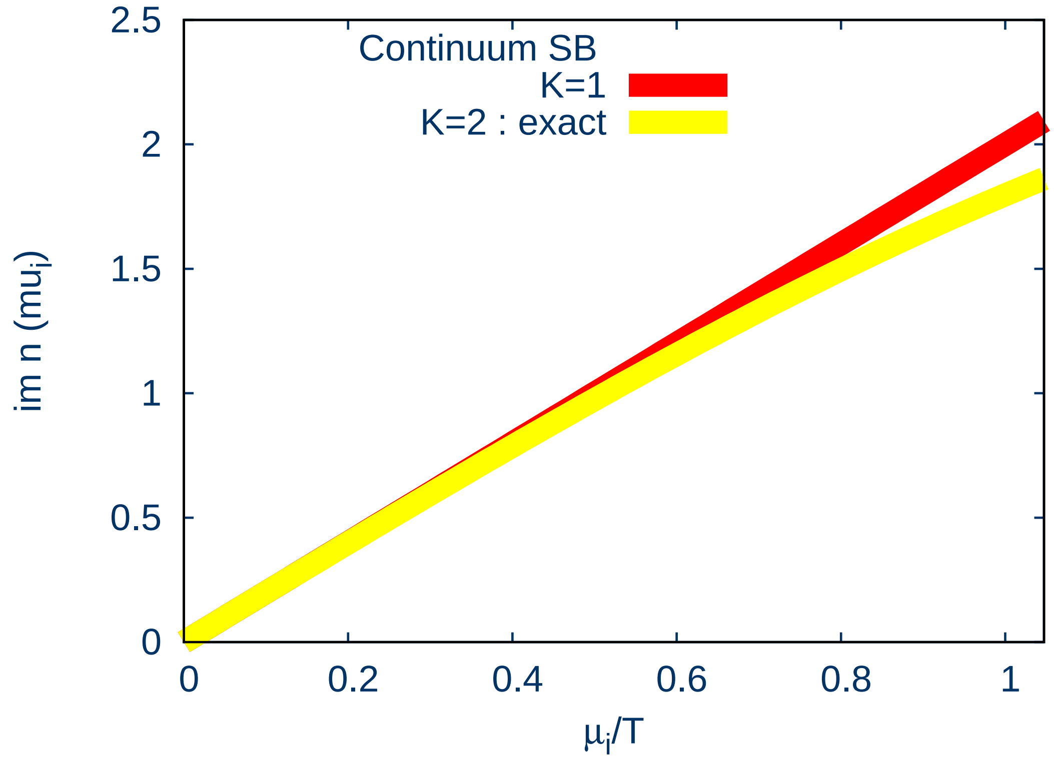
$$n_q(T,\mu_q)/T^3 = lim_{K\to\infty} n_q^K(T,\mu_q)/T^3$$

Stefan-Boltzmann : $n_q^{(2)}(T,\mu_q)$ exact

$$n_{SB}/T^3 = 2(\mu/T) + \frac{2}{\pi^2}(\mu/T)^3$$

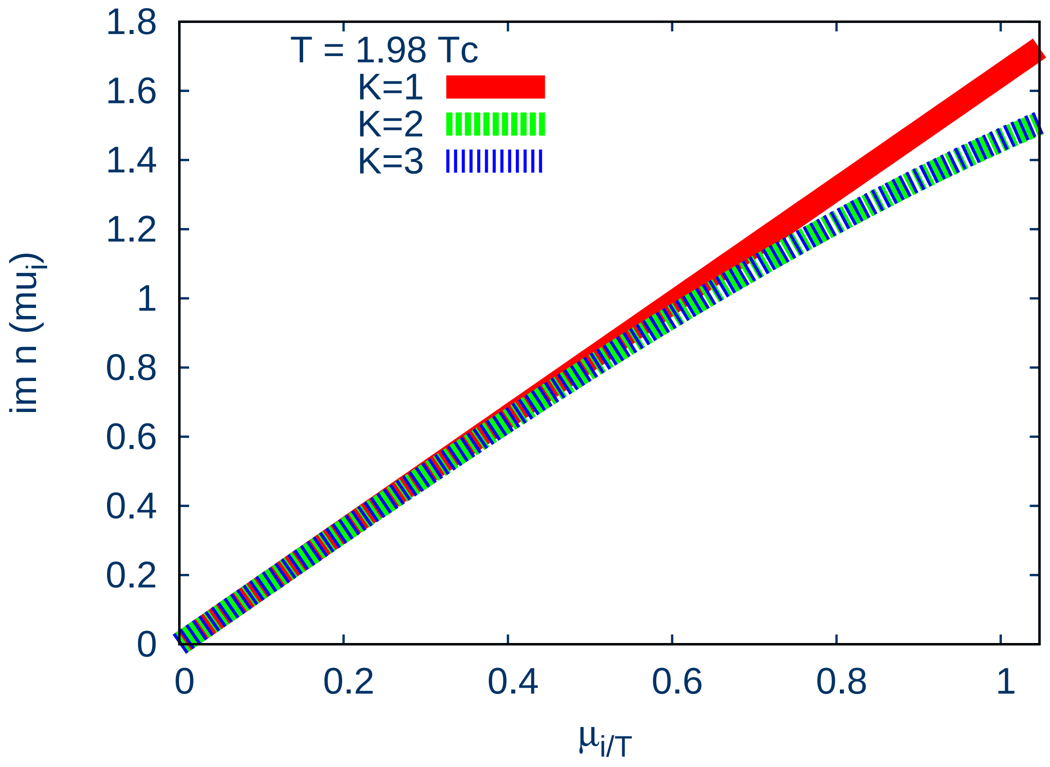
As a function of μ_I:

$$Im(n_{SB})/T^3 = 2(\mu_I/T) - 2/(\pi^2)(\mu_I/T)^3$$

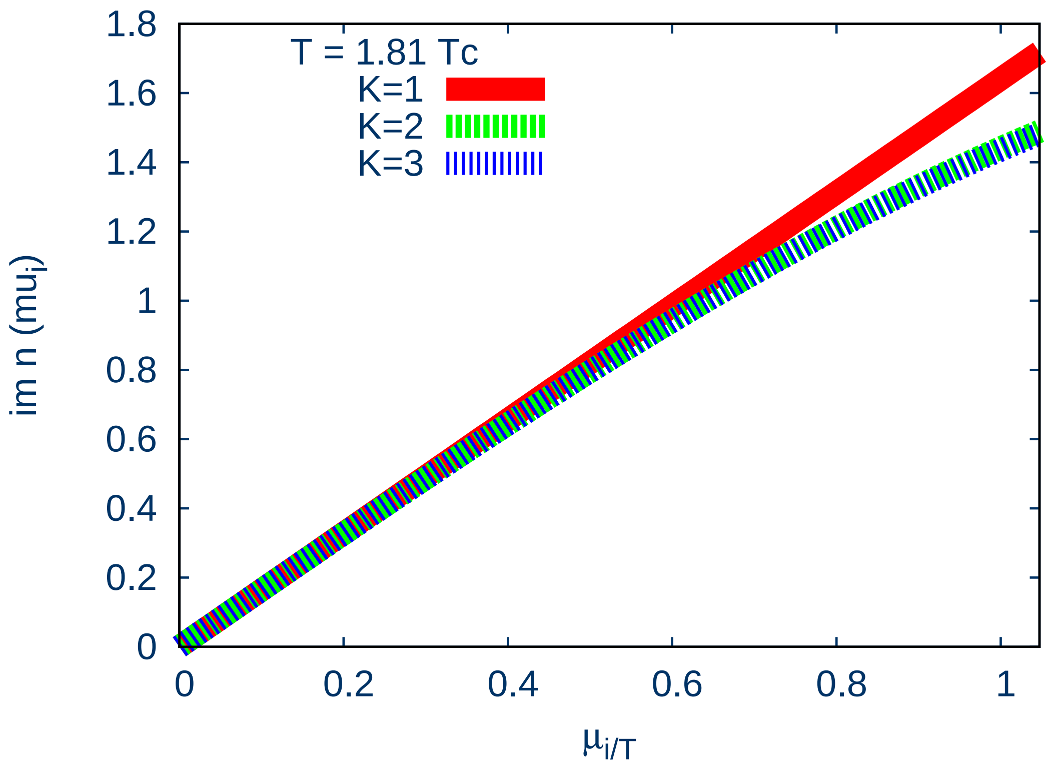


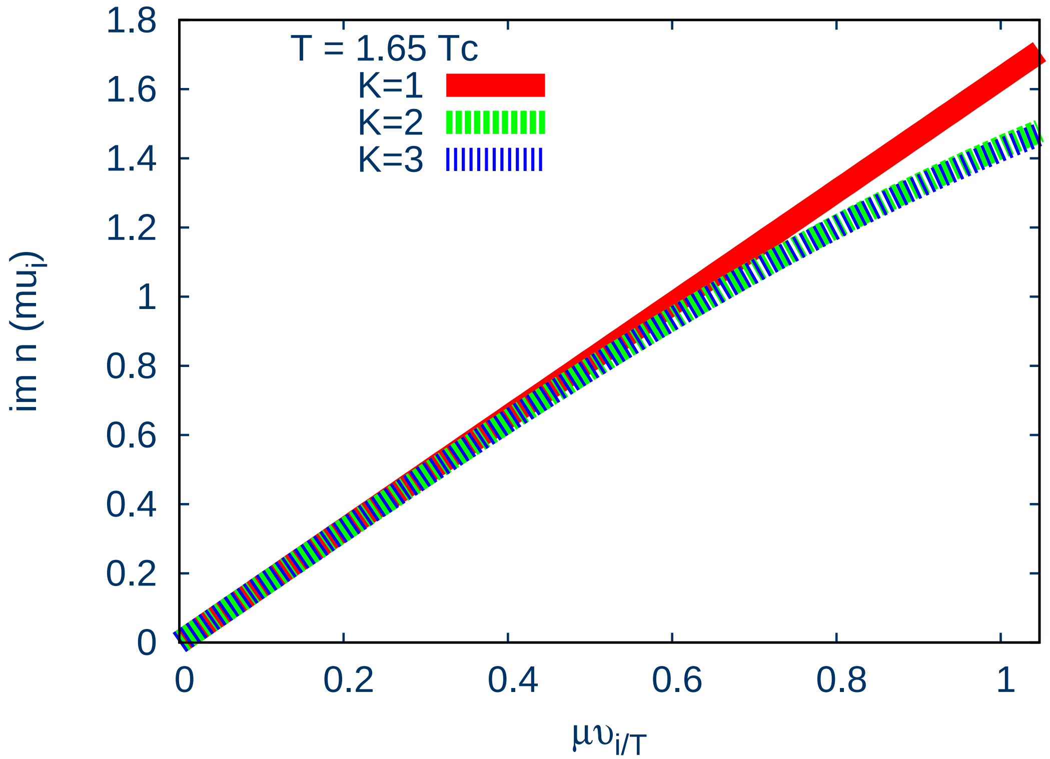
$Im(n_q^1(T,\mu_I))/T^3$ and $Im(n_q^2(T,\mu_I))/T^3$ (exact)
in the Stefan-Boltzmann limit

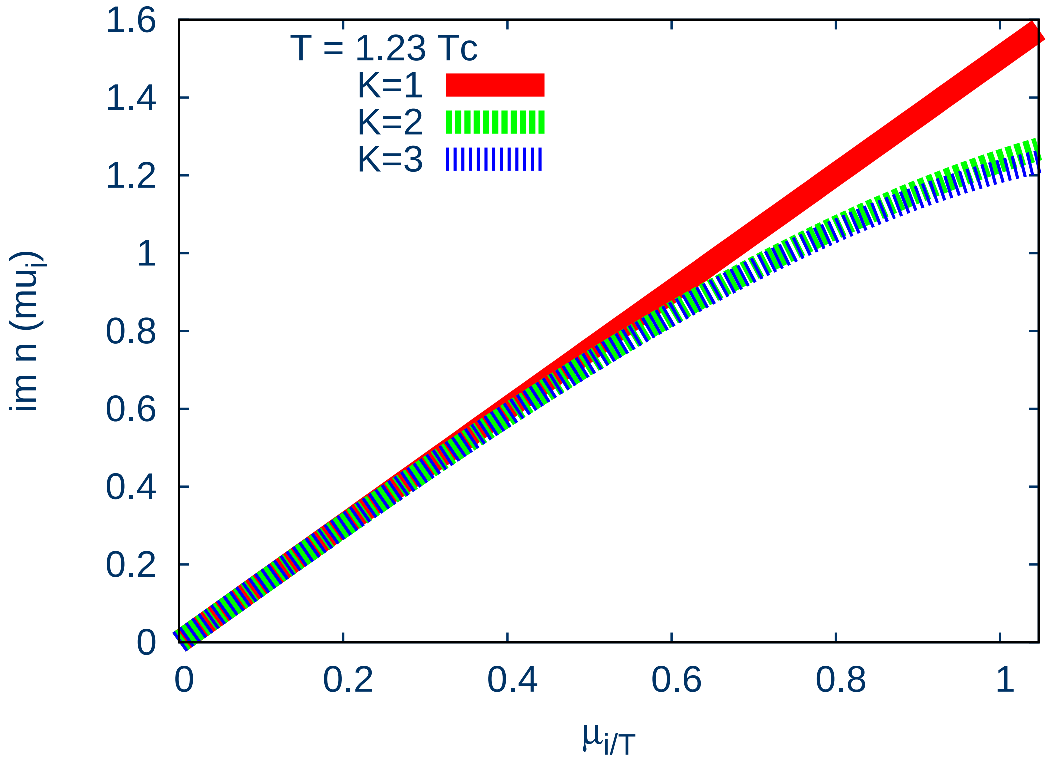
FROM STEFAN BOLTZMANN TO sQGP AT μ IMAGINARY

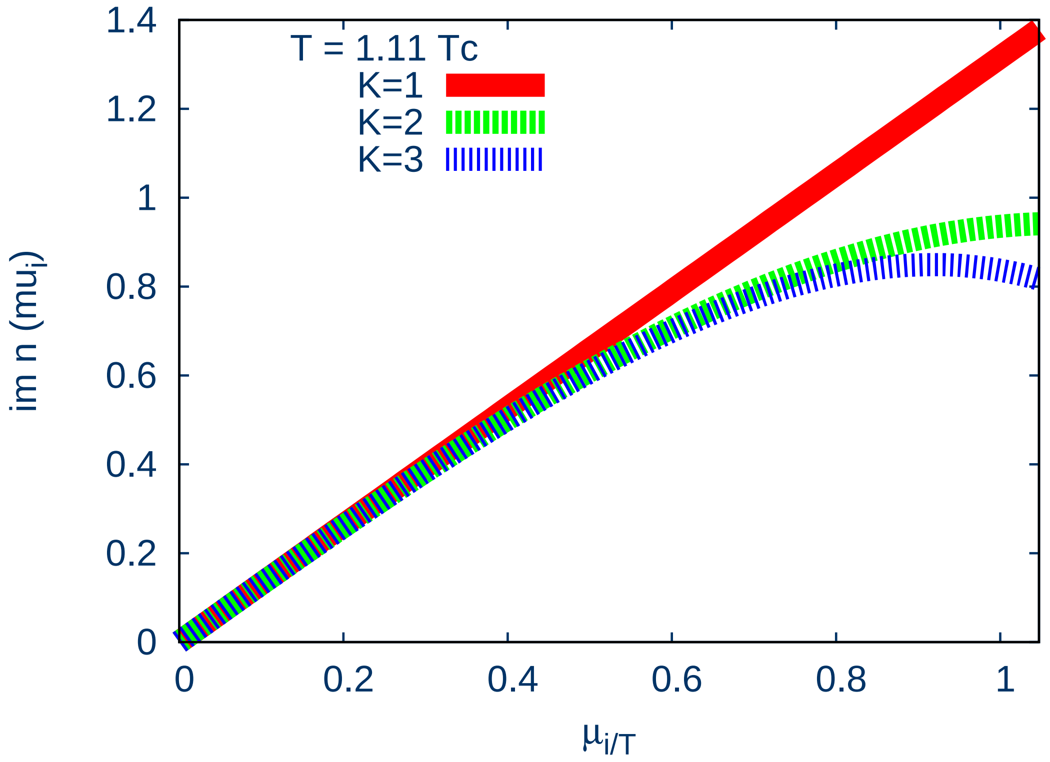


T = 1.81 Tc : converges everywhere

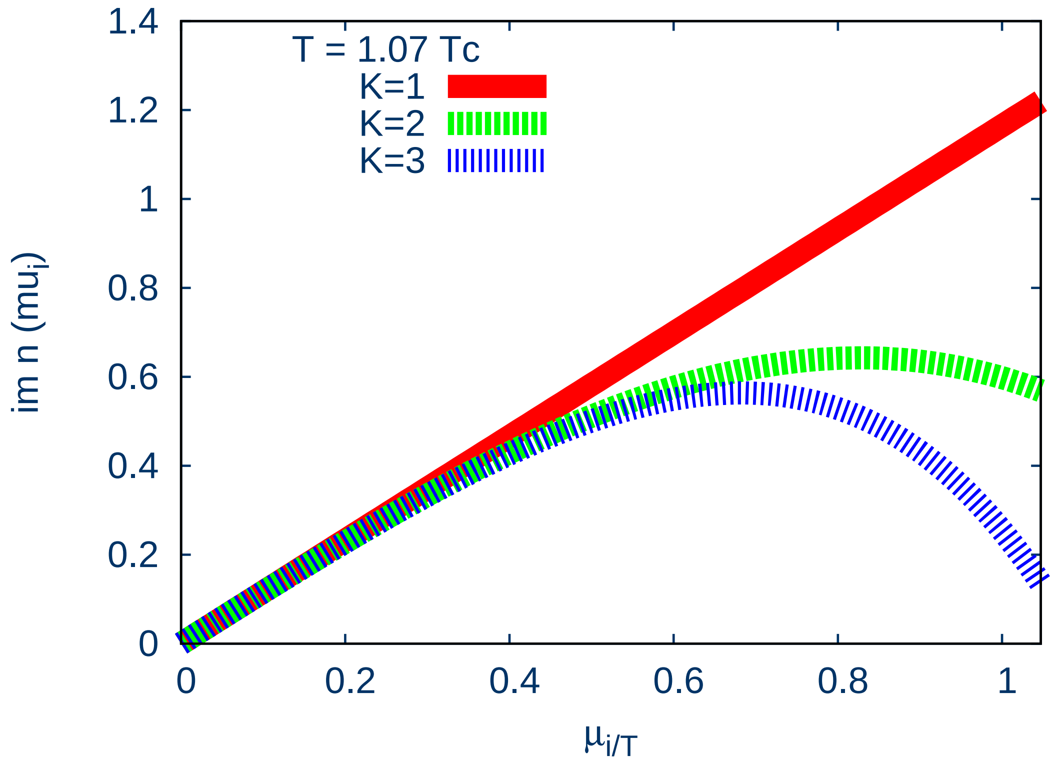



T = 1.65 Tc
K=1
K=2
K=3
im n (mu$_i$)
μυ$_{i/T}$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

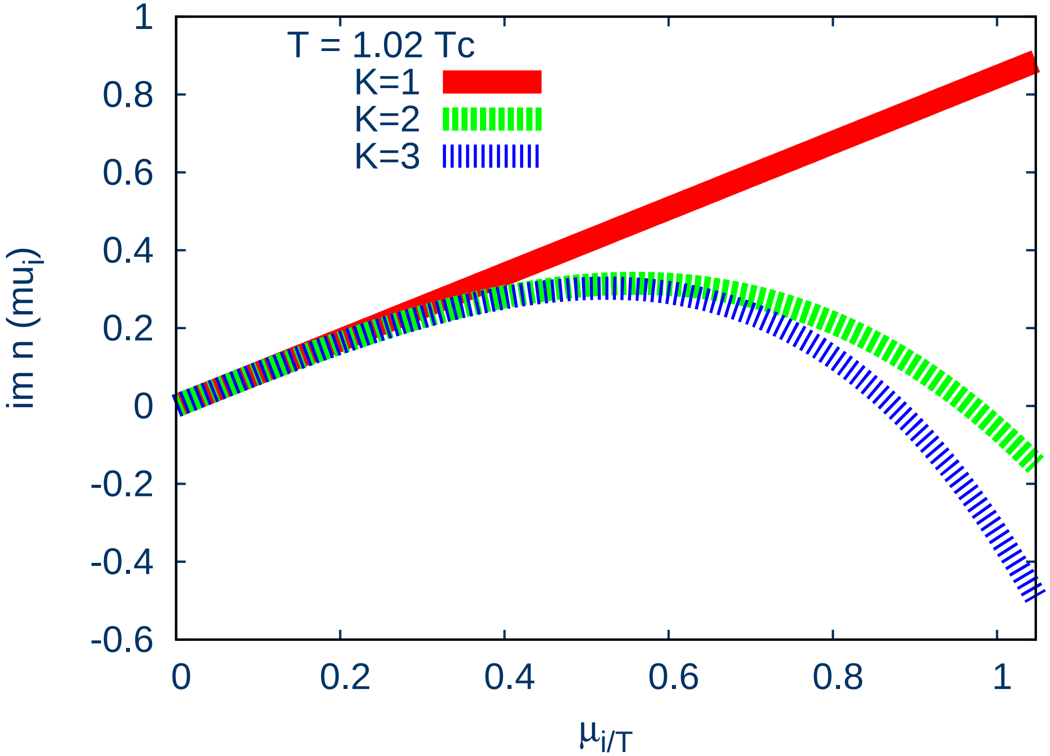

T = 1.23 Tc
K=1
K=2
K=3
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
im n (mu_i)
0
0.2
0.4
0.6
0.8
1
μ_i/T


T = 1.11 Tc
K=1
K=2
K=3
im n (mu$_i$)
μ$_{i/T}$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

T = 1.07 Tc : CONVERGES FOR $\mu_I < 0.6$



T = 1.07 Tc : converges for $\mu_I < 0.4$



For $T > 1.1T_c$ the results From the Taylor expansion nicely achieve convergence in the interval of periodicity. Radius of convergence is infinite.

At lower temperatures the first three coefficients do not suffice to achieve convergence.

The role of higher order coefficients, which are very difficult to compute, becomes important there.

That suggests that imaginary μ calculations can help filling the gap and complete the information of $n(\mu)$ in the available interval $[0 : \pi/3]$

Radius of convergence might well be limited by the critical line at imaginary μ.

THE RW ENDPOINT AT μ_I FOR $N_f = 2$

The Roberge Weiss line is at $\mu/T = \pi/3$, extending from $T = \infty$ till $T = T_{RW}$. T_{RW} was estimated in the four flavor model $T_{RW} = 1.1T_c$.

We can make a crude estimate of the location of the Roberge Weiss endpoint in two flavor by remembering that the slope of the critical line depends linearly on N_f, concluding that T_{RW} in the two flavor model should be $\simeq 1.05T_c$.

In conclusion, for this model, in the high T phase, imaginary chemical potential calculations would complement the Taylor expansion results for $T_c < T < 1.1T_c, \mu_I/T > 0.4$. This would allow an assessment of the critical behaviour associated with the RW endpoint.

One qualitative argument for this is the possible role of residual baryonic excitations, which become important at larger chemical potential in the strongly coupled QGP phase.

THE HADRONIC PHASE $T < T_c$

The natural framework to parametrize the results in the low temperature domain is the hadron resonance gas model in the Boltzmann limit

$$n_q^{HG}(T, \mu_q)/T^3 = F(T) sinh(3\mu_q/T)$$

For purely imaginary $\mu_q = i\mu_I$

$$n_q^{HG}(T, \mu_q)/T^3 = F(T) sin(3\mu_q/T)$$

FREEZOUT

Values of μ_q^F/T at freezout for the temperatures used in the lattice simulations.

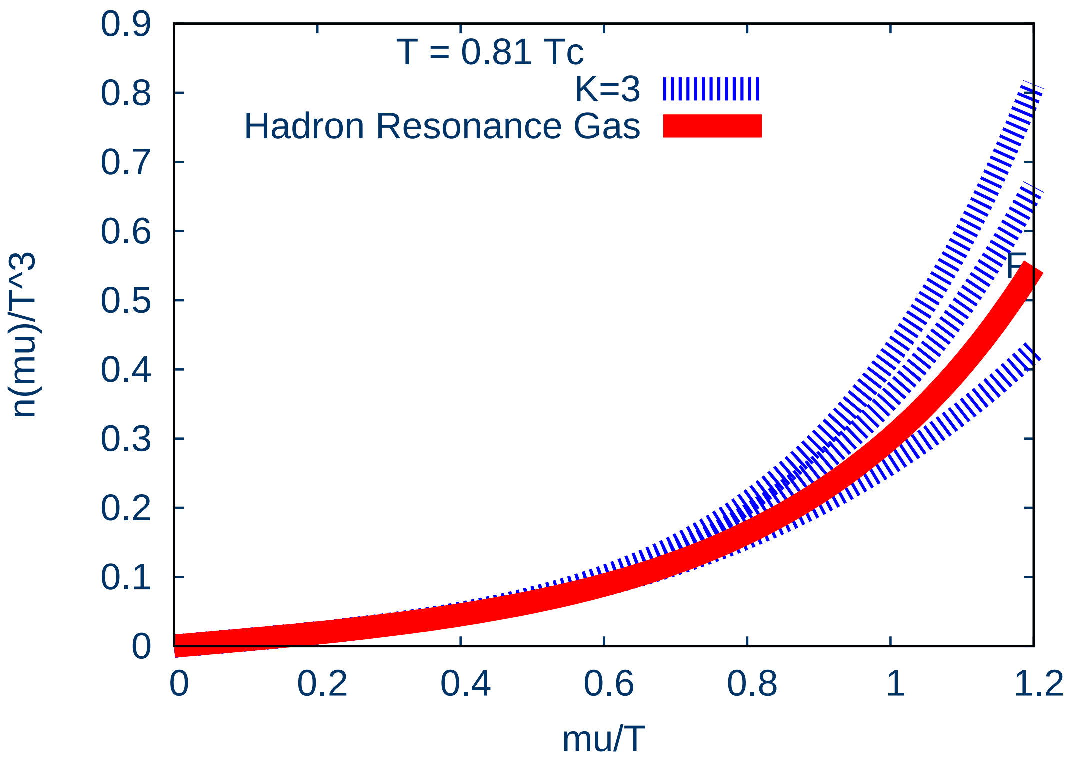
Table 1: Freezout parameters

T/T_c	μ_B^F (GeV)	μ_q^F/T
0.81	0.48	1.16
0.87	0.38	0.85
0.90	0.3	0.65
0.96	0.15	0.30

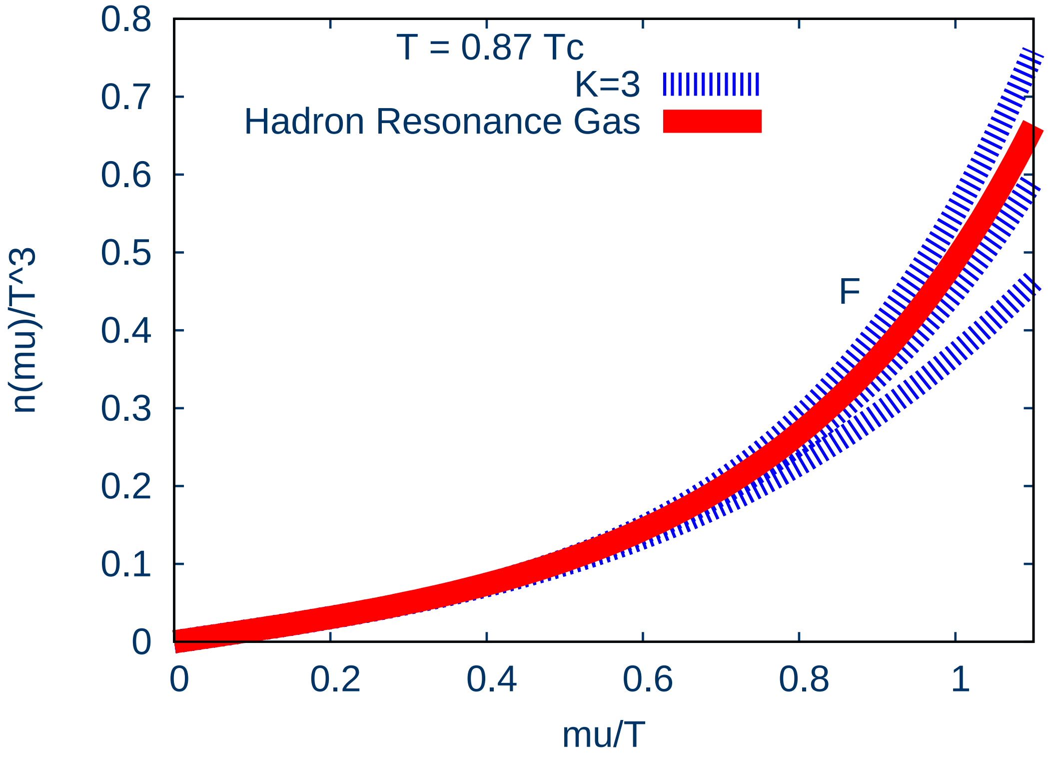
Previous analysis have shown that for this range of temperatures the Hadron Gas parametrization is satisfied by the first coefficents.

Then, to assess the extent of the convergence, we can directly contrast $n_q^3(T,\mu_{q_I})/T^3$ and $n_q^{HG}(T,\mu_{q_I})/T^3$, with $F(T) = \frac{2}{3}c_2$.

T = 0.81 Tc

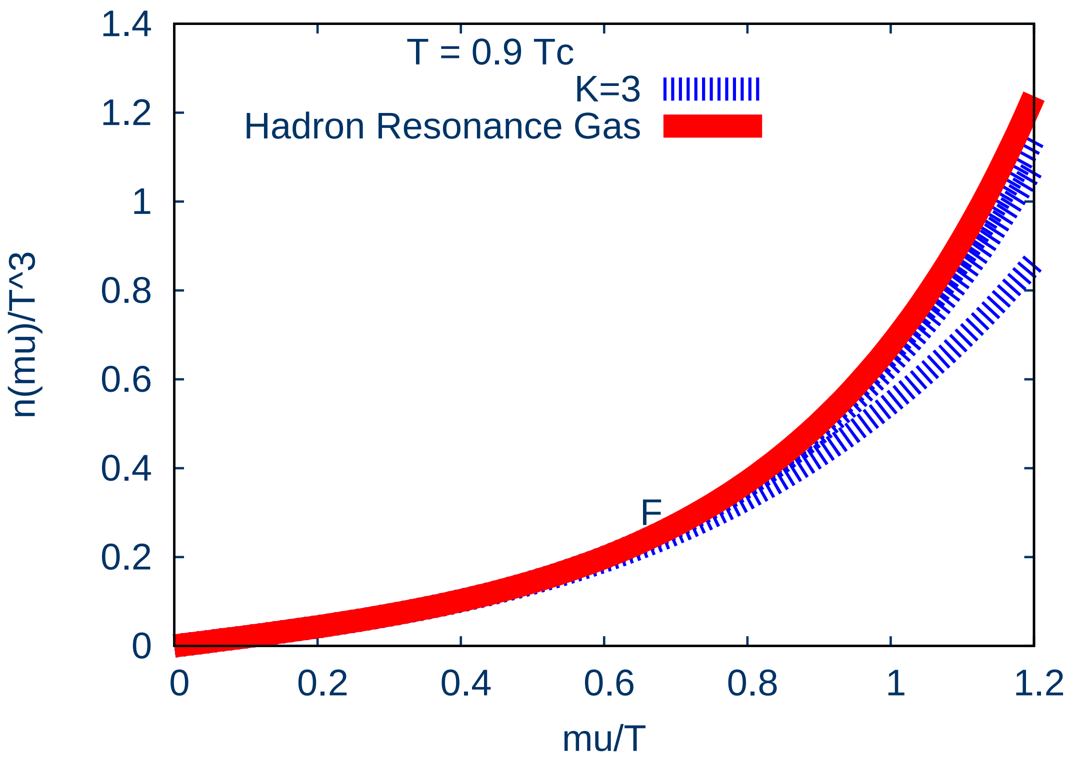

T = 0.81 Tc
K=3
Hadron Resonance Gas
n(mu)/T^3
mu/T
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2

T = 0.87 Tc

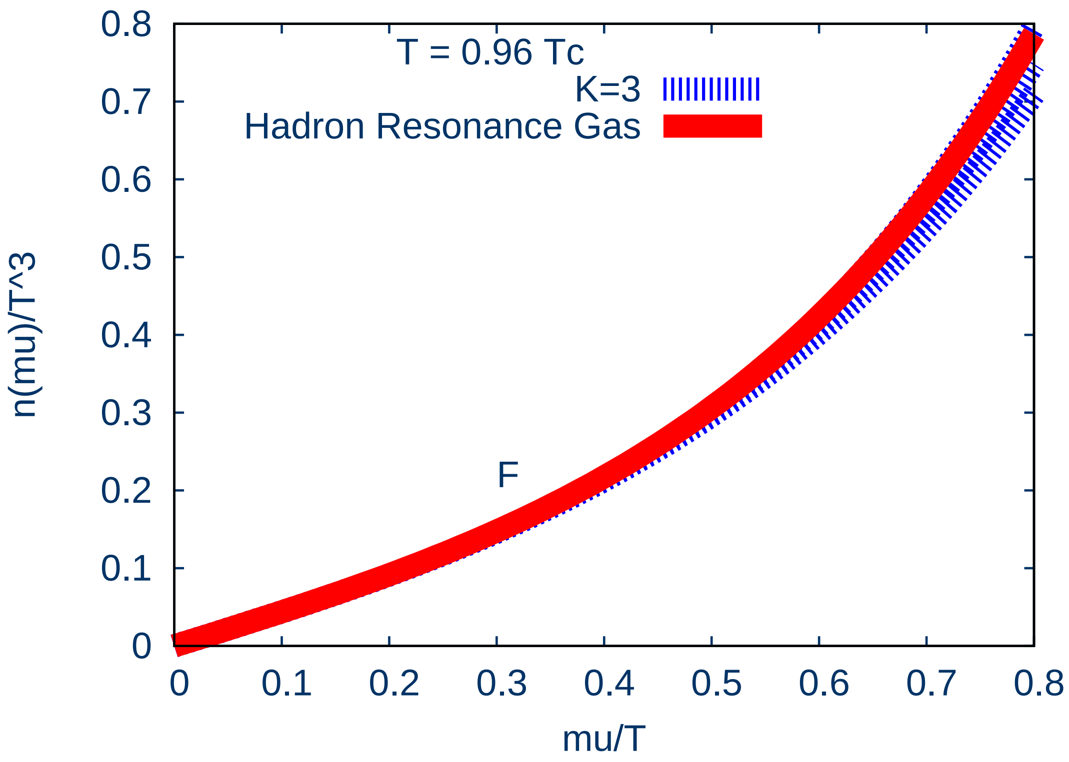


T = 0.87 Tc
K=3
Hadron Resonance Gas
F
n(mu)/T^3
mu/T
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
0.2
0.4
0.6
0.8
1

T = 0.90 Tc



T = 0.96 Tc



T = 0.96 Tc
K=3
Hadron Resonance Gas
F
n(mu)/T^3
mu/T
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Critical behaviour and radius of convergence . I

$T > T_c$

We can with a good confidence rule out a possible critical behaviour associated with the RW singularity for $T > 1.1T_c$, where we have observed a good convergence to a simple polynomial form.

In the strongly coupled QGP region our crude estimate of the Roberge Weiss end-point is $T^E_{RW} \simeq 1.05T_c$, which is consistent with the observed pattern of convergence

We can also place a lower bound on the radius of convergence r in μ/T, namelly $r > 0.3$ for $T = 1.02T_c$, $r > 0.4$ for $T = 1.07T_c$, $r > 0.6$ for $T = 1.11T_c$.

CRITICAL BEHAVIOUR AND RADIUS OF CONVERGENCE . II

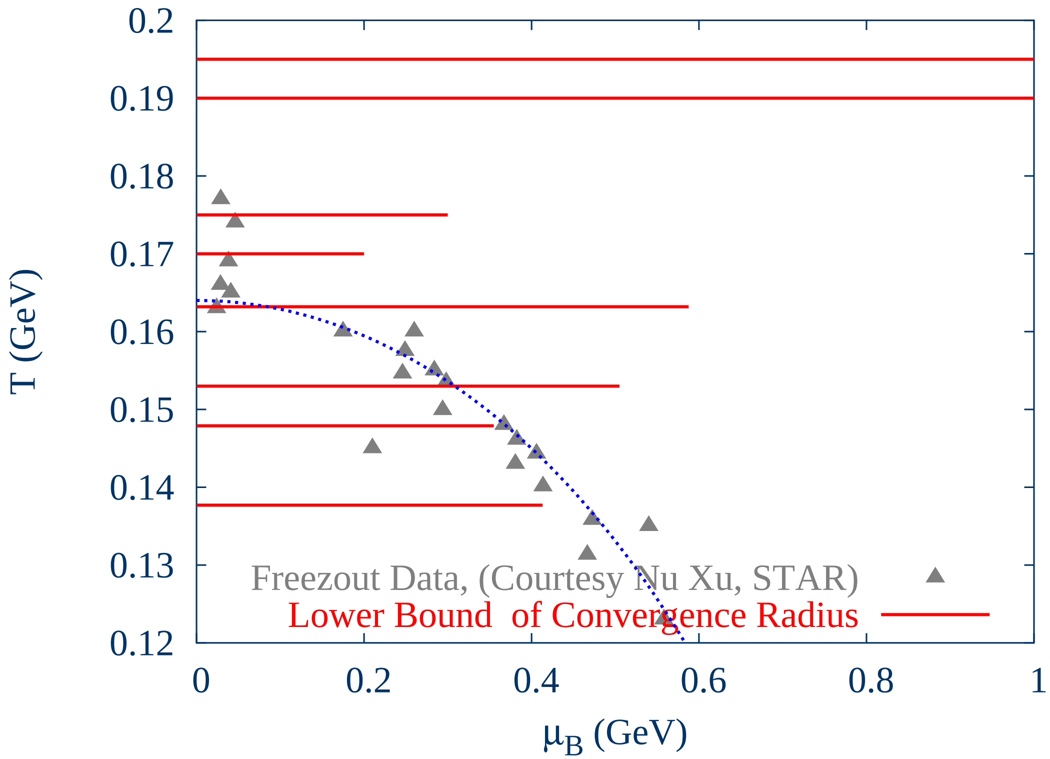
$T < T_c$

In this region we expect an analytic behaviour till the critical line (possibly larger for a strong first order transition)

In the crossover region the radius of convergence might grow large

The freezout region is apparently contained inside the convergence domain, which makes it accessible to a numerical analysis.

These conclusions are systematically improvable in a controlled way



0.2
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
T (GeV)
0
0.2
0.4
0.6
0.8
1
μB (GeV)
Freezout Data, (Courtesy Nu Xu, STAR)
Lower Bound of Convergence Radius

SUMMARY

- Extrapolation methods have
  - Produced results for the slope of the critical line and thermodynamics
  - Triggered strategies for the endpoint
  - Motivated analysis of QCD in a complex parameter space
- Combined application of imaginary chemical potential and Taylor expansion seems able to give:
  - Reliable results for the radius of convergence in the complex plane and indications for the endpoint of QCD (or lack thereof)
  - Thermodynamics results in a region of interest for FAIR/RHIC II around freezout.