Symmetries, signs, and complex actions in lattice effective field theory

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ECT* Workshop on Signs and Complex Actions
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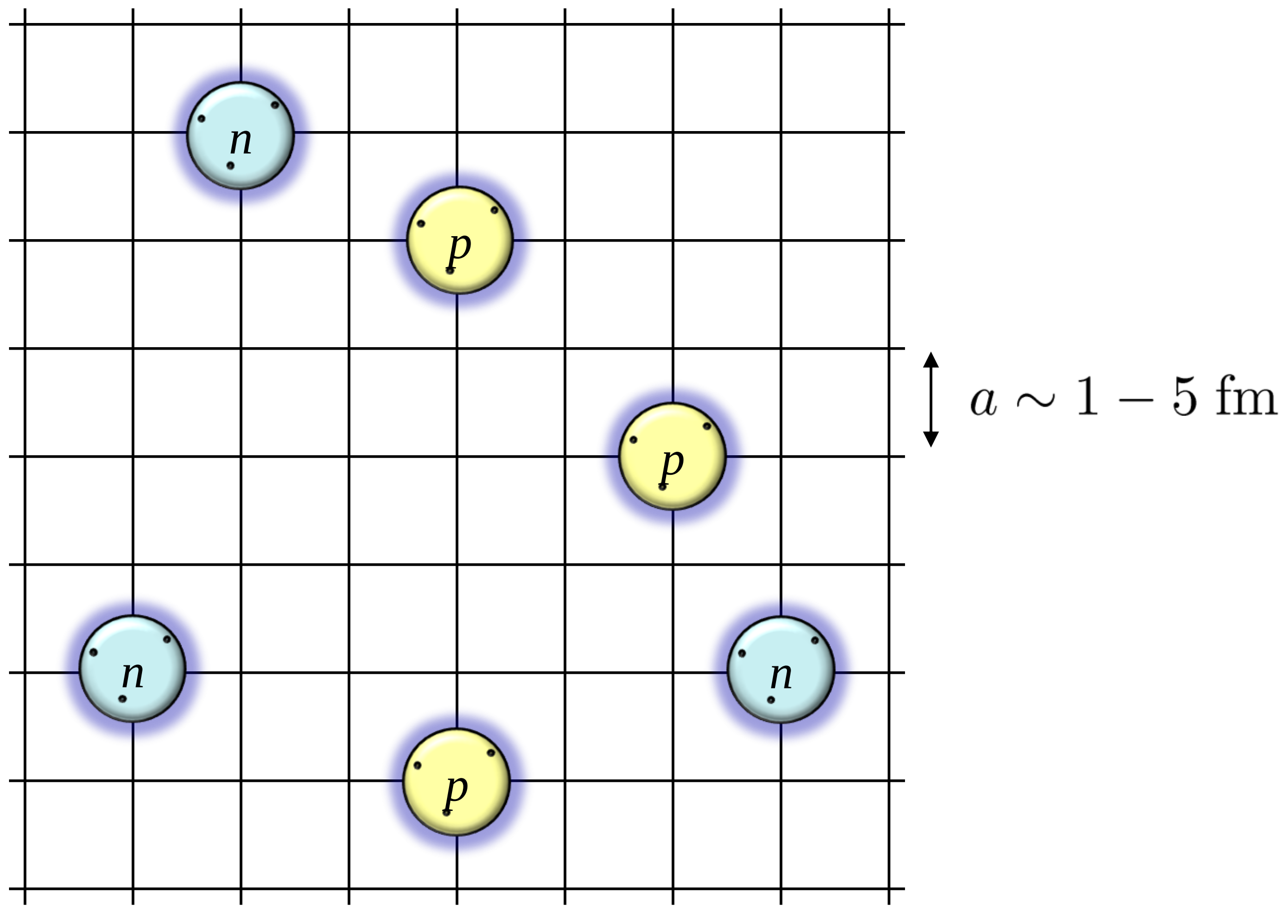
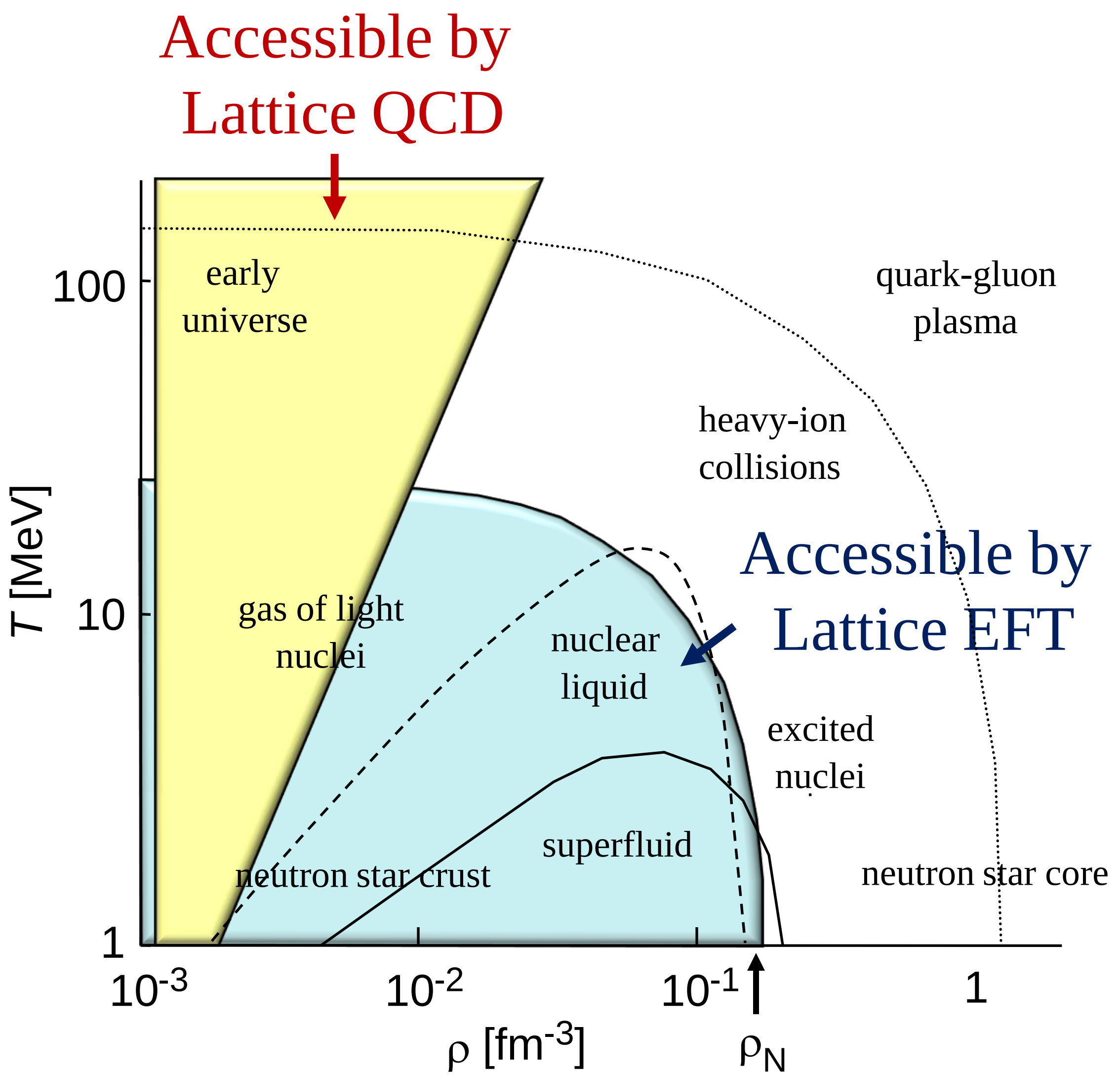
Outline

- What is lattice effective field theory?
- Chiral effective field theory for nucleons
- Computational strategies on the lattice
- Auxiliary fields, signs, and complex actions
- Phase shifts and unknown operator coefficients
- Dilute neutron matter at NLO
- How severe is the sign problem really?
- Studies of light nuclei at NNLO
- Summary, future directions, and connections

Lattice EFT for nucleons



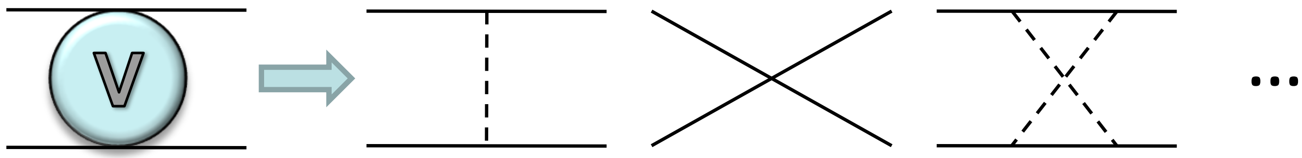


Accessible by
Lattice QCD
early
universe
quark-gluon
plasma
heavy-ion
collisions
gas of light
nuclei
nuclear
liquid
Accessible by
Lattice EFT
excited
nuclei
superfluid
neutron star crust
neutron star core
100
10
1
T [MeV]
10-3
10-2
10-1
1
ρ [fm-3]
ρN

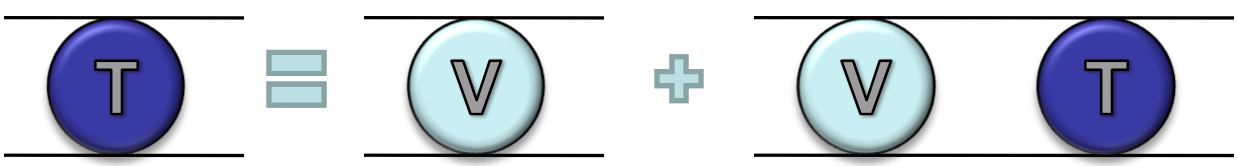
Chiral EFT for low-energy nucleons

Weinberg, PLB 251 (1990) 288; NPB 363 (1991) 3

Construct the effective potential order by order



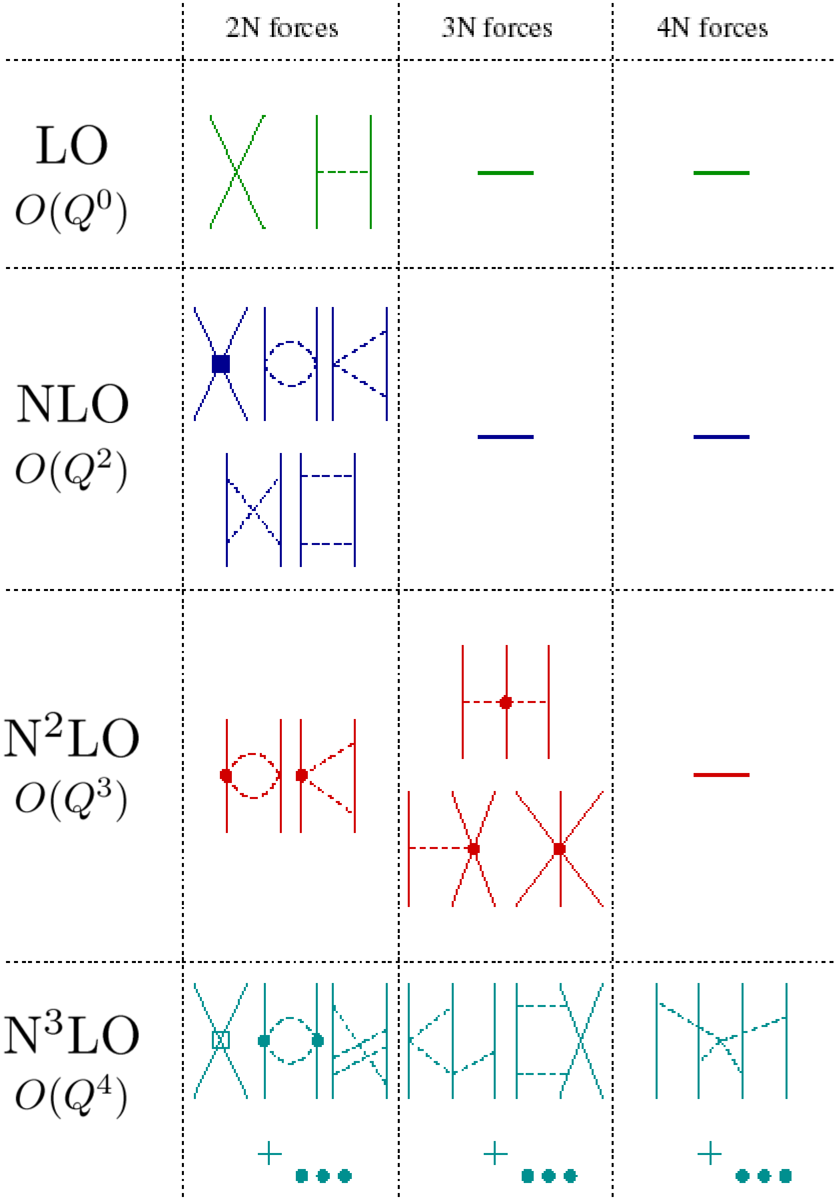
Solve Lippmann-Schwinger equation non-perturbatively



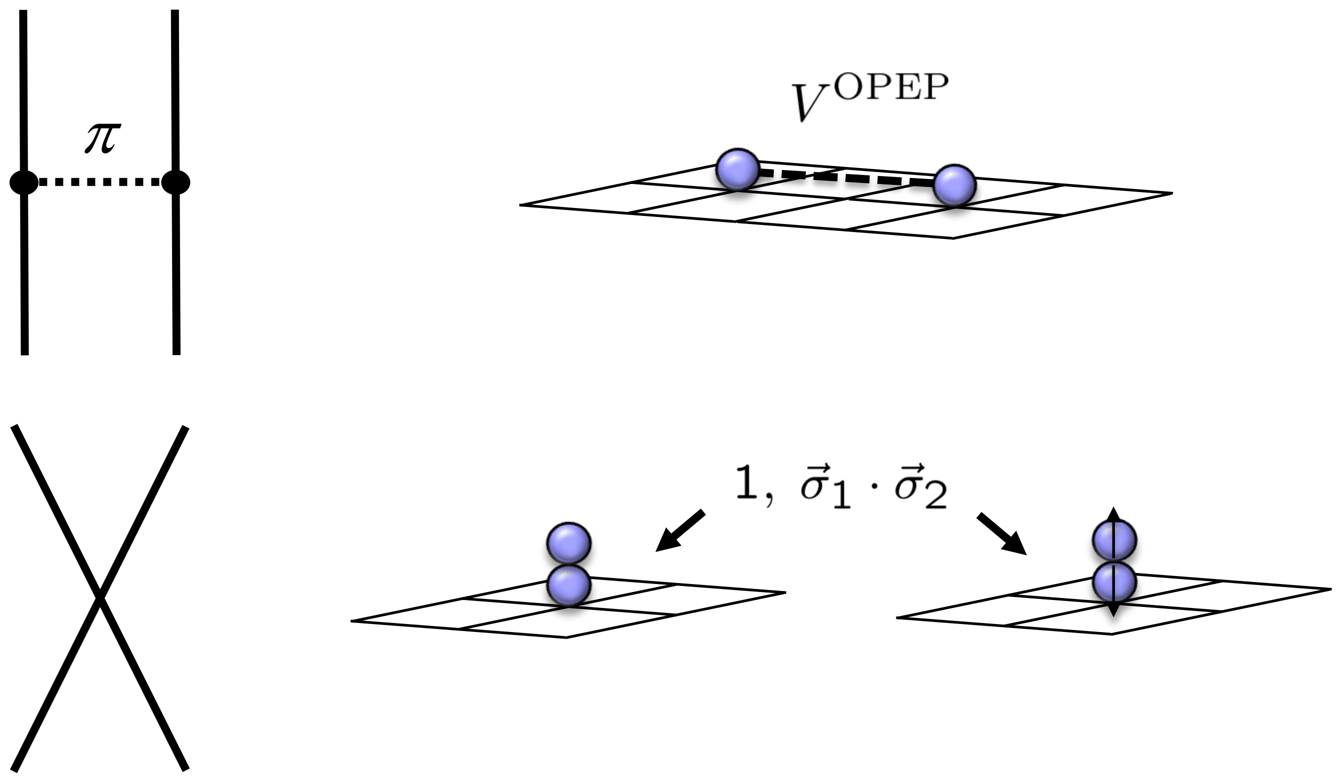
Nuclear Scattering Data

Effective Field Theory

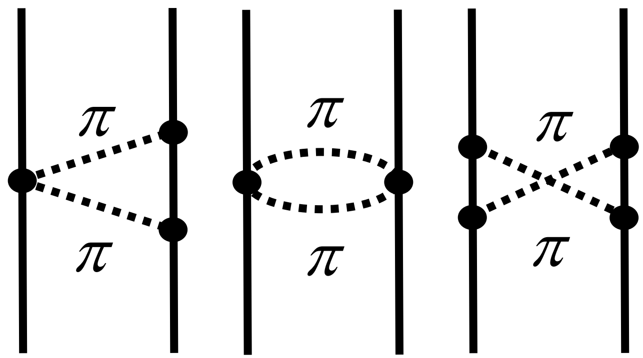
Ordonez et al. ’94; Friar & Coon ’94; Kaiser et al. ’97; Epelbaum et al. ’98, ‘03; Kaiser ’99-’01; Higa et al. ’03; …



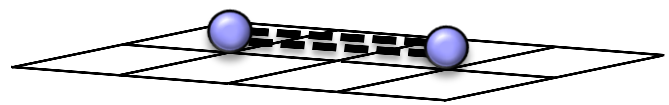
Leading order on lattice

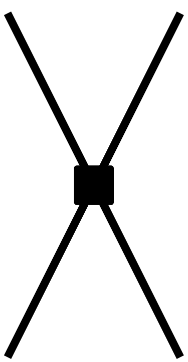


Next-to-leading order on lattice

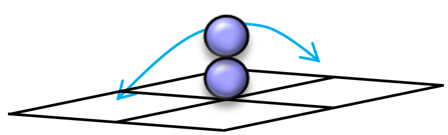


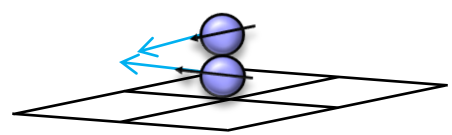
V^{TPEP}



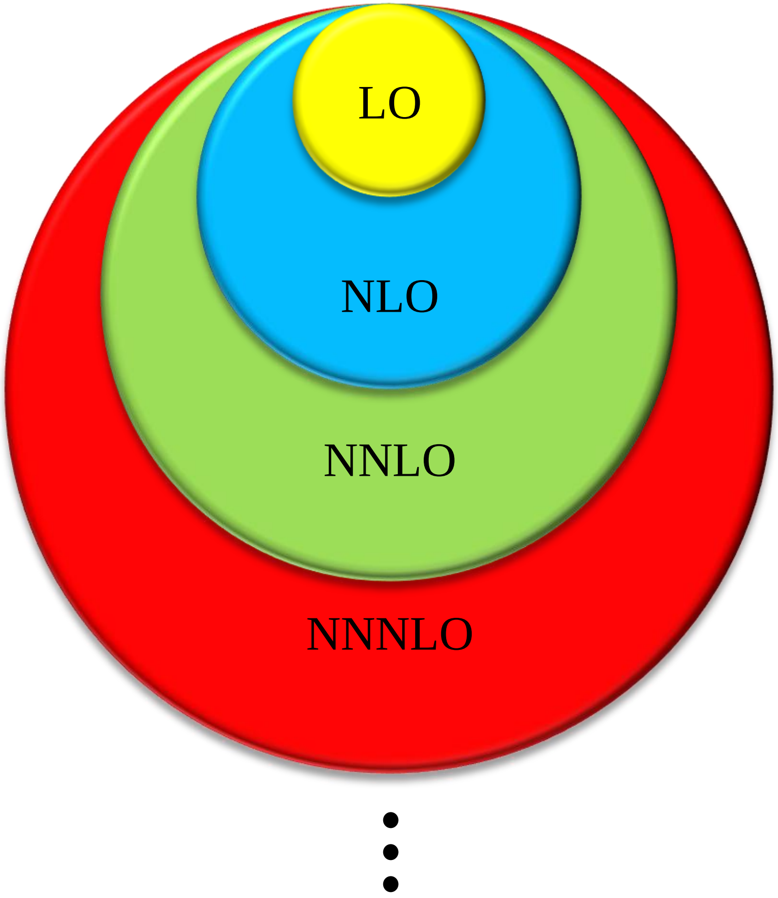


$\vec{\nabla}_1 \cdot \vec{\nabla}_2 \qquad (\vec{\sigma}_1 \cdot \vec{\nabla}_1)\,(\vec{\sigma}_2 \cdot \vec{\nabla}_2) \qquad \cdots$

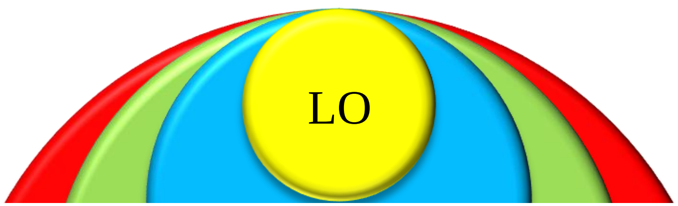




Computational strategy

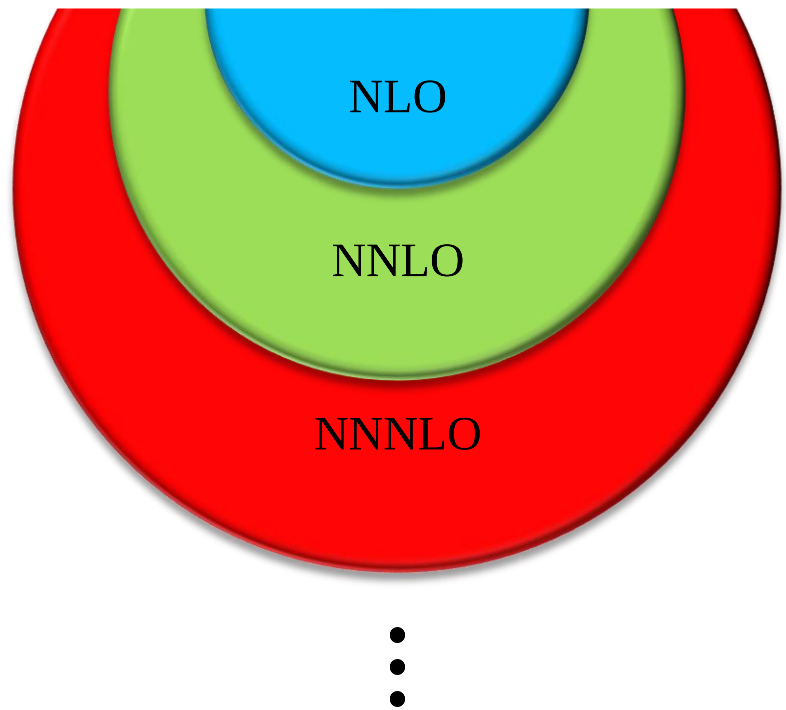


Non-perturbative – Monte Carlo



“Improved LO”

Perturbative corrections



Lattice formulations

Grassmann path integral $\exp\left[-S(c, c^*)\right]$	$=$	**Transfer matrix operator** M	$\xrightarrow[a_t \to 0]{}$	**Hamiltonian operator** H
$\Vert$		$\Vert$		$\Vert$
Grassmann path integral with auxiliary/pion field $\exp\left[-S_A(c, c^*, \pi, s)\right]$	$=$	**Transfer matrix operator with auxiliary/pion field** $M_A(s, \pi, n_t)$	$\xrightarrow[a_t \to 0]{}$	**Hamiltonian operator with auxiliary/pion field** $M_A(s, \pi, t)$

Euclidean-time transfer matrix

Free nucleons: $$\exp\left[\frac{1}{2m}N^\dagger\vec{\nabla}^2 N\Delta t\right]$$

Free pions: $$\exp\left[-\frac{1}{2}\left(\vec{\nabla}\boldsymbol{\pi}\right)^2\Delta t-\frac{m_\pi^2}{2}\boldsymbol{\pi}^2\Delta t\right]$$

Pion-nucleon coupling: $$\exp\left[-\frac{g_A}{2f_\pi}N^\dagger\boldsymbol{\tau}\vec{\sigma}N\cdot\cdot\vec{\nabla}\boldsymbol{\pi}\Delta t\right]$$

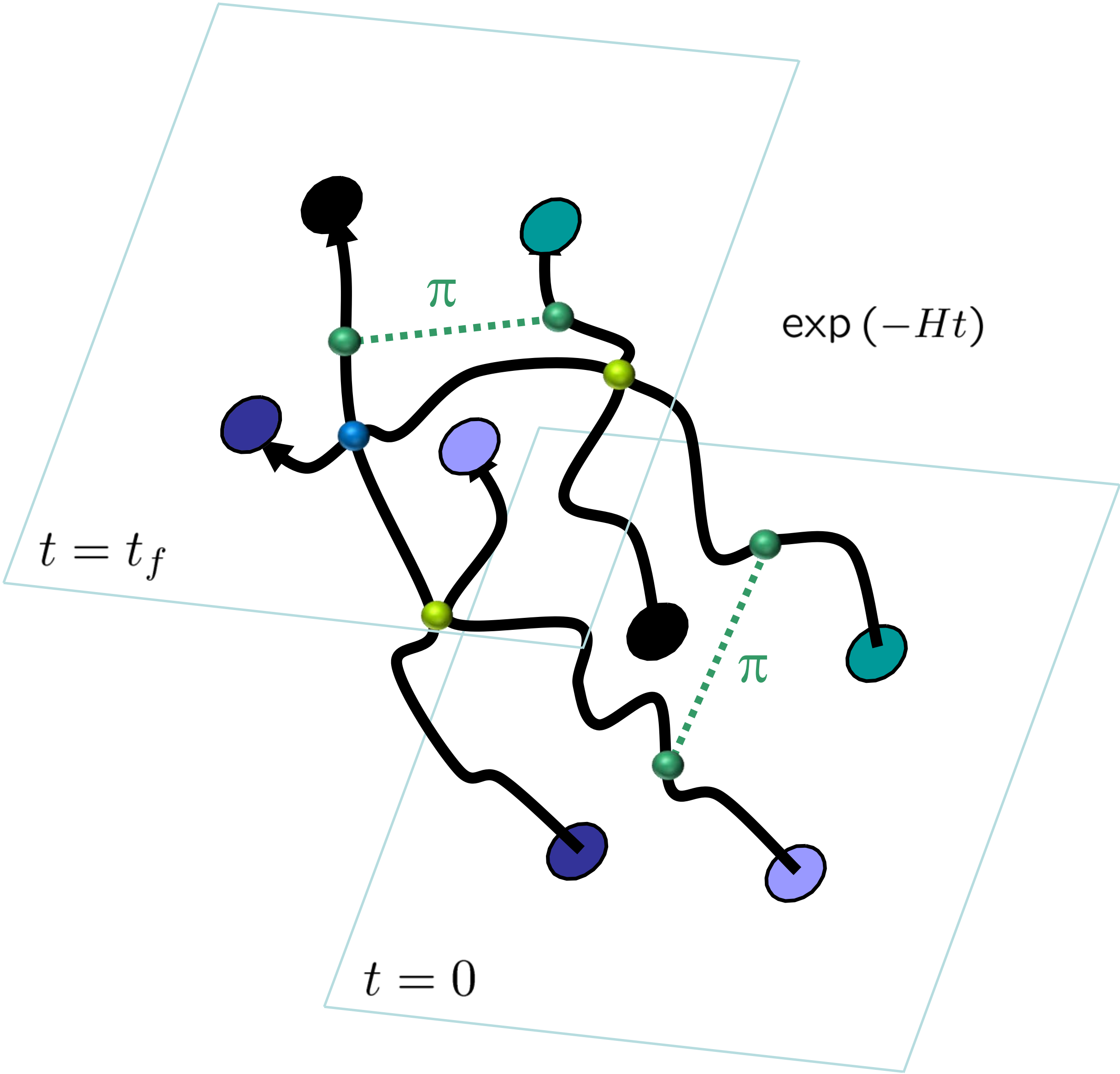
… with auxiliary fields

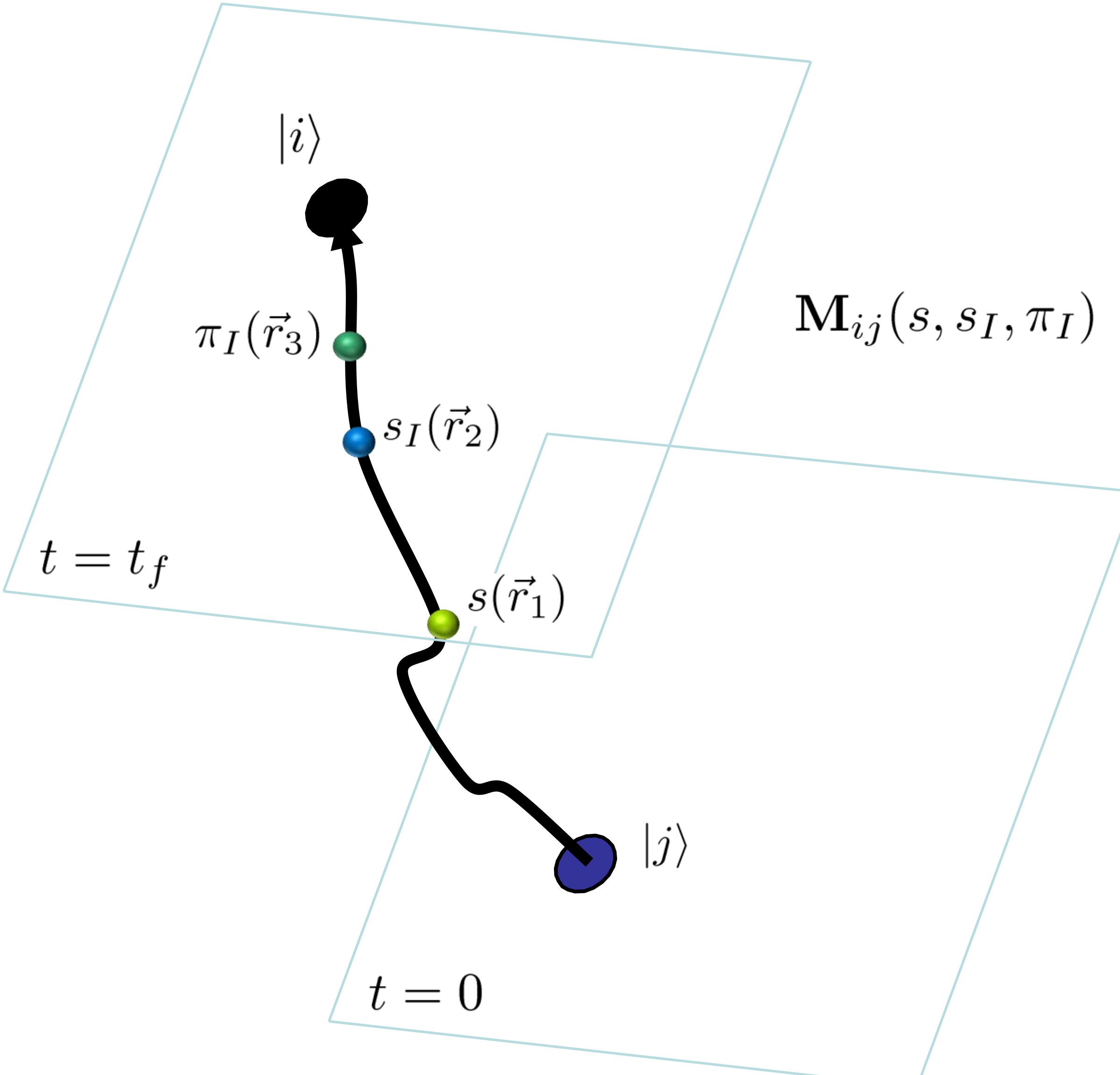
C contact interaction:

$$\exp\left[-\frac{1}{2}CN^\dagger N N^\dagger N \Delta t\right] \quad (C<0)$$
$$= \frac{1}{\sqrt{2\pi}}\int ds \exp\left[-\frac{1}{2}s^2 + sN^\dagger N\sqrt{-C\Delta t}\right]$$

C_I contact interaction:

$$\exp\left[-\frac{1}{2}C_I N^\dagger \boldsymbol{\tau} N \cdot N^\dagger \boldsymbol{\tau} N \Delta t\right] \quad (C_I>0)$$
$$= \frac{1}{\sqrt{2\pi}}\int d\boldsymbol{s}_I \exp\left[-\frac{1}{2}\boldsymbol{s}_I \cdot \boldsymbol{s}_I + i\boldsymbol{s}_I \cdot N^\dagger \boldsymbol{\tau} N\sqrt{C_I\Delta t}\right]$$


π
$\exp(-Ht)$
$t = t_f$
π
$t = 0$


$|i\rangle$
$\pi_I(\vec{r}_3)$
$\mathbf{M}_{ij}(s, s_I, \pi_I)$
$s_I(\vec{r}_2)$
$t = t_f$
$s(\vec{r}_1)$
$|j\rangle$
$t = 0$

Euclidean-time projection Monte Carlo

$$\langle\psi_{\text{init}}|\,M^{(L_t-1)}(s,s_I,\pi_I)\cdot\cdots\cdot M^{(0)}(s,s_I,\pi_I)\,|\psi_{\text{init}}\rangle = \det\mathbf{M}(s,s_I,\pi_I)$$

$$\mathbf{M}_{ij}(s,s_I,\pi_I) = \langle\vec{p}_i|\,M^{(L_t-1)}(s,s_I,\pi_I)\cdots M^{(0)}(s,s_I,\pi_I)\,|\vec{p}_j\rangle$$

For *A* nucleons, the matrix is *A* by *A*. For the leading-order calculation, if there is no pion coupling and the quantum state is an isospin singlet then

$$\tau_2\mathbf{M}\tau_2 = \mathbf{M}^*$$

This shows the determinant is real. Actually we can show that the determinant is positive semi-definite

Consider an eigenvector

$$\mathbf{M}\phi = \lambda\phi$$

Let us define a new vector

$$\tilde{\phi} = \tau_2\phi^*$$

Note that

$$\begin{aligned}\mathbf{M}\tilde{\phi} &= \mathbf{M}\tau_2\phi^* = \tau_2\tau_2\mathbf{M}\tau_2\phi^* = \tau_2\mathbf{M}^*\phi^* \\ &= \tau_2\left(\mathbf{M}\phi\right)^* = \tau_2\left(\lambda\phi\right)^* = \lambda^*\tau_2\phi^* = \lambda^*\tilde{\phi}\end{aligned}$$

Note also that the two vectors are orthogonal

$$\tilde{\phi}^\dagger\phi = \left(\tau_2\phi^*\right)^\dagger\phi = \phi^T\tau_2^\dagger\phi = \phi^T\tau_2\phi = 0$$

So complex eigenvalues come in conjugate pairs, and the real spectrum is doubly-degenerate

With nonzero pion coupling the determinant is real for a spin-singlet isospin-singlet quantum state

$$\sigma_2\tau_2\mathbf{M}\sigma_2\tau_2 = \mathbf{M}^*$$

but the determinant can be both positive and negative

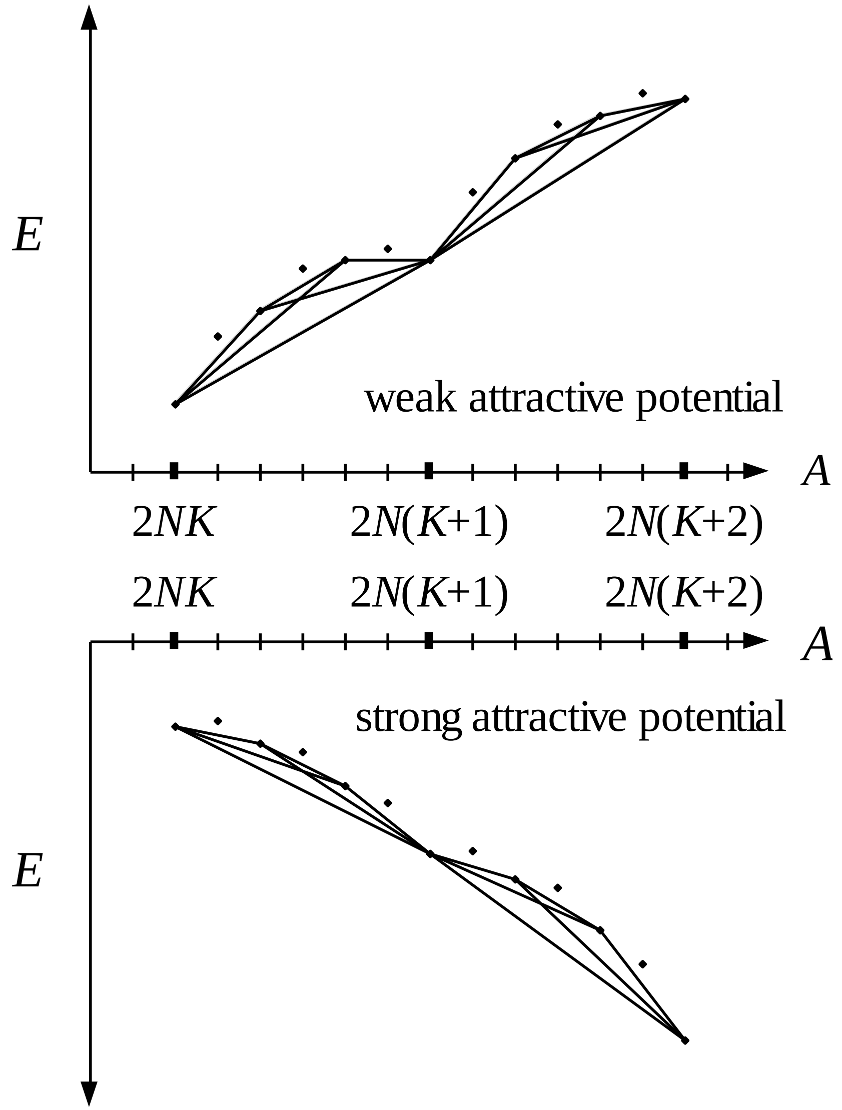
Some comments about Wigner's approximate SU(4) symmetry…

Theorem: Any fermionic theory with SU(2N) symmetry and two-body potential with negative semi-definite Fourier transform

$$\tilde{V}(\vec{p}) \cdot \quad 0$$

satisfies SU(2N) convexity bounds…

SU(2N) convexity bounds



$$|\psi_{\rm init}\rangle \to \{j \text{ flavors occupation } n_1, 2N-j \text{ flavors occupation } n_2\}$$

$$\langle\psi_{\rm init}|\, M^{(L_t-1)}(\phi)\cdot\cdots\cdot M^{(0)}(\phi)\,|\psi_{\rm init}\rangle = \det \mathbf{M}(\phi)$$

$$\int D\phi\, e^{-S(\phi)}\ \det\mathbf{M}(\phi)$$

$$S(\phi) = -\frac{\alpha_t}{2}\sum_{n_t}\sum_{\vec{n},\vec{n}'}\phi(\vec{n},n_t)V^{-1}(\vec{n}-\vec{n}')\phi(\vec{n}',n_t)$$

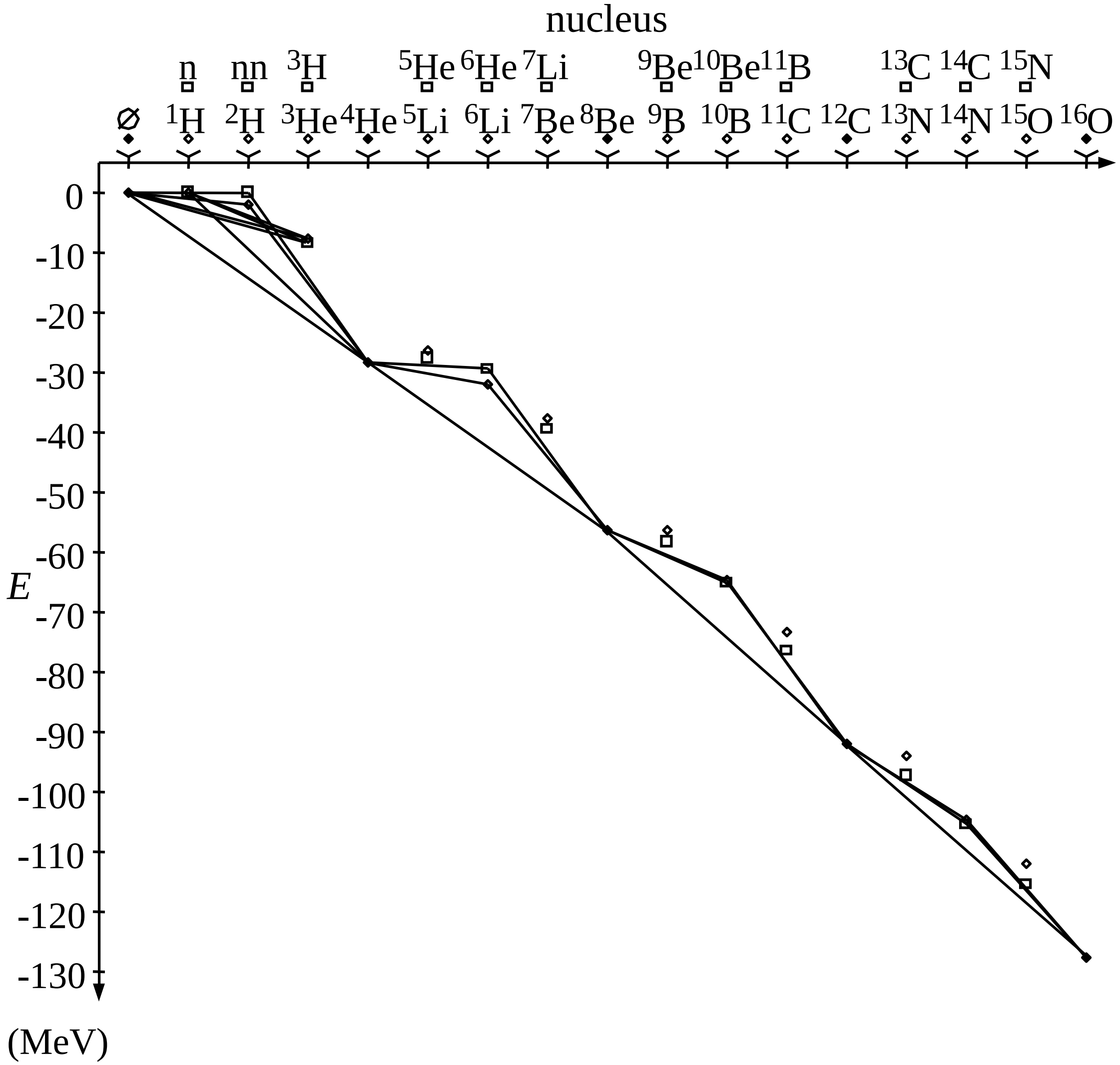
$$\mathbf{M}_{k',k}(\phi) = \langle f_{k'}|\, M^{(L_t-1)}(\phi)\times\cdots\times M^{(0)}(\phi)\,|f_k\rangle$$

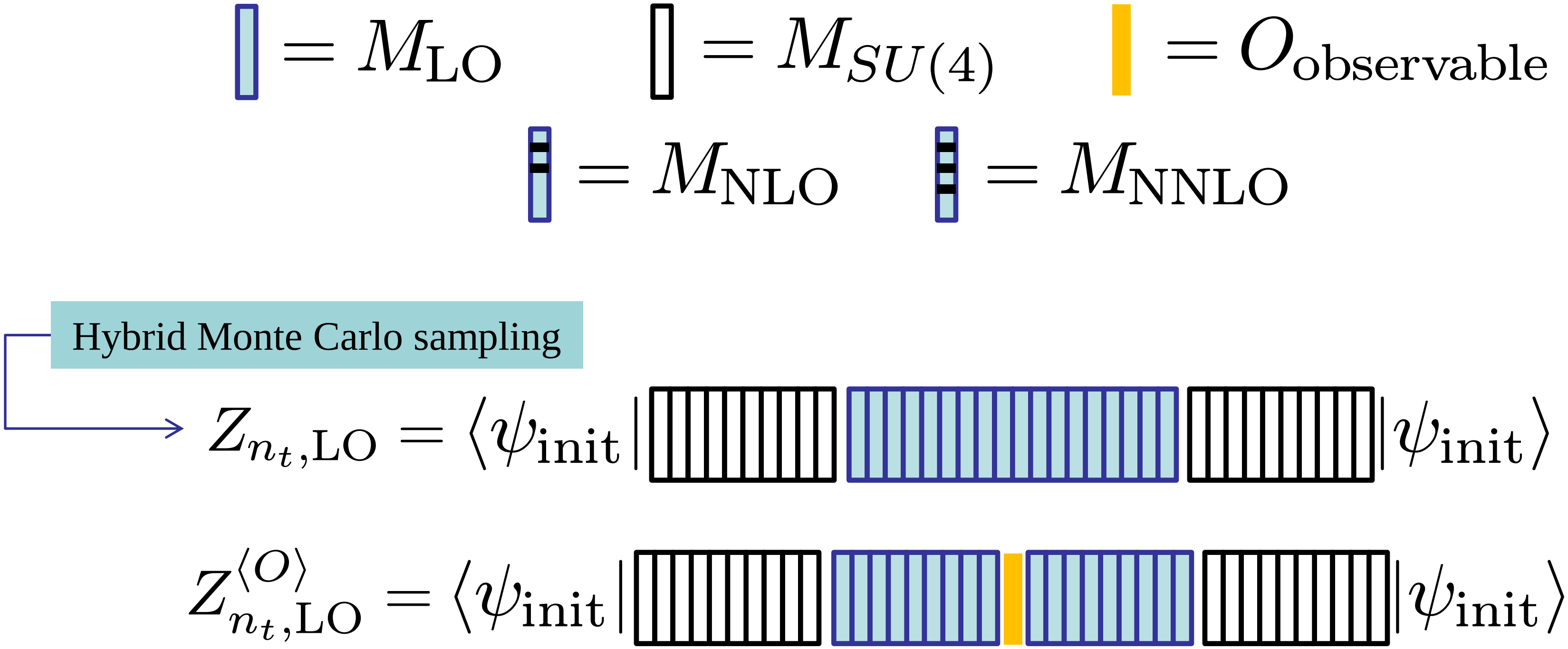
$$\int D\phi\, e^{-S(\phi)}\ \left[\det\mathbf{M}_{n_1\times n_1}(\phi)\right]^j \left[\det\mathbf{M}_{n_2\times n_2}(\phi)\right]^{2N-j}$$

Apply Hölder inequality

Chen, D.L. Schäfer, PRL 93 (2004) 242302;
D.L., PRL 98 (2007) 182501

SU(4) convexity bounds

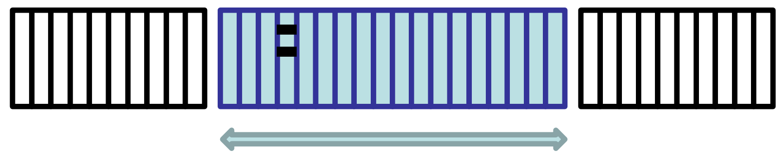


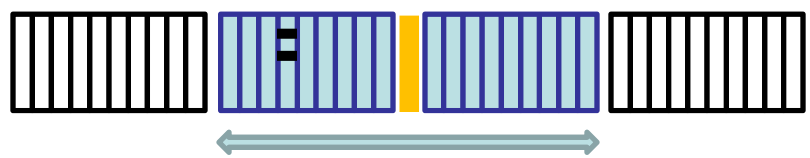


$$e^{-E_{0,\mathrm{LO}} a_t} = \lim_{n_t \to \infty} Z_{n_t+1,\mathrm{LO}} / Z_{n_t,\mathrm{LO}}$$

$$\langle O \rangle_{0,\mathrm{LO}} = \lim_{n_t \to \infty} Z^{\langle O \rangle}_{n_t,\mathrm{LO}} / Z_{n_t,\mathrm{LO}}$$

$$(1 - \Delta E_{0,\mathrm{NLO}} a_t) e^{-E_{0,\mathrm{LO}} a_t} = \langle M_{\mathrm{NLO}} \rangle_{0,\mathrm{LO}}$$

$Z_{n_t,\mathrm{NLO}} = \langle \psi_{\mathrm{init}} |$  $| \psi_{\mathrm{init}} \rangle$

$Z^{\langle O \rangle}_{n_t,\mathrm{NLO}} = \langle \psi_{\mathrm{init}} |$  $| \psi_{\mathrm{init}} \rangle$

$$\langle O \rangle_{0,\mathrm{NLO}} = \lim_{n_t \to \infty} Z^{\langle O \rangle}_{n_t,\mathrm{NLO}} / Z_{n_t,\mathrm{NLO}}$$

LO_1: Pure contact interactions

$$\mathcal{A}(V_{\mathrm{LO}_1}) = C + C_I \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + \mathcal{A}(V^{\mathrm{OPEP}})$$

LO_2: Gaussian smearing

$$\mathcal{A}(V_{\mathrm{LO}_2}) = C f(\vec{q}^{\,2}) + C_I f(\vec{q}^{\,2}) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + \mathcal{A}(V^{\mathrm{OPEP}})$$

LO_3: Gaussian smearing only in even partial waves

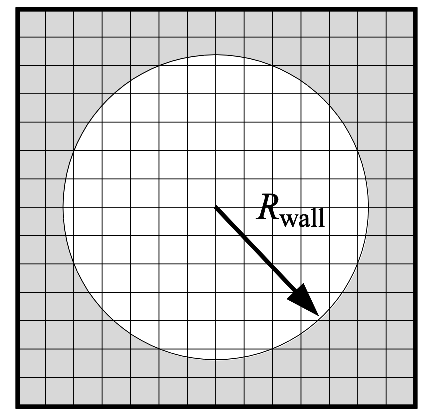
$$\begin{aligned}\mathcal{A}(V_{\mathrm{LO}_3}) = {} & C_{1S0} f(\vec{q}^{\,2}) \left(\tfrac{1}{4} - \tfrac{1}{4}\vec{\sigma}_1 \cdot \vec{\sigma}_2\right) \left(\tfrac{3}{4} + \tfrac{1}{4}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\right) \\ & + C_{3S1} f(\vec{q}^{\,2}) \left(\tfrac{3}{4} + \tfrac{1}{4}\vec{\sigma}_1 \cdot \vec{\sigma}_2\right) \left(\tfrac{1}{4} - \tfrac{1}{4}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\right) \\ & + \mathcal{A}(V^{\mathrm{OPEP}})\end{aligned}$$

Physical scattering data → Unknown operator coefficients

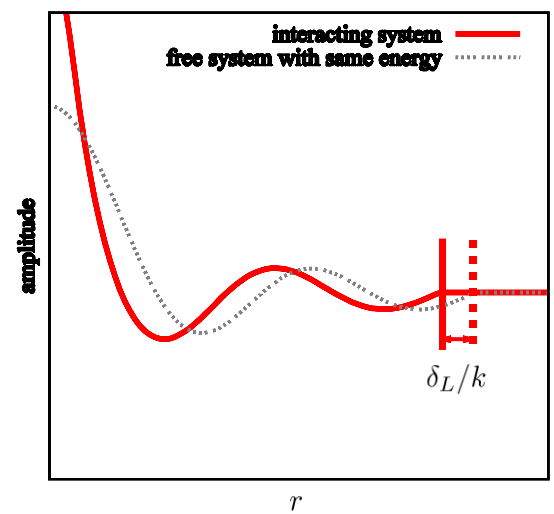
Spherical wall method

Borasoy, Epelbaum, Krebs, D.L., Meißner,
EPJA 34 (2007) 185

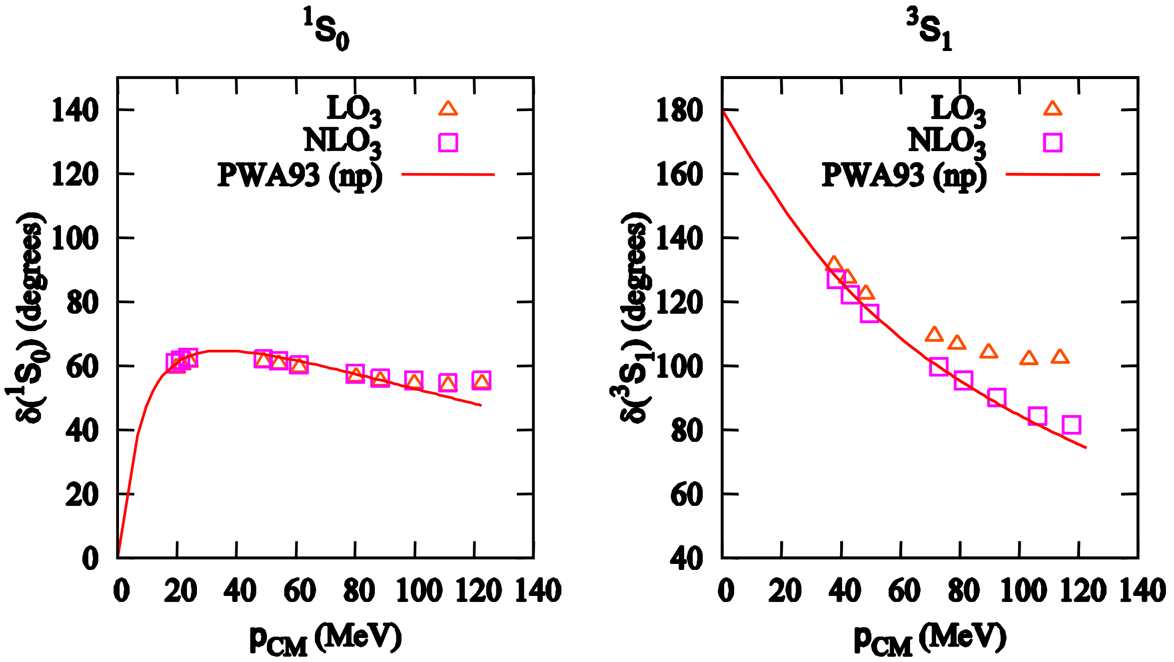
Spherical wall imposed in the center of mass frame



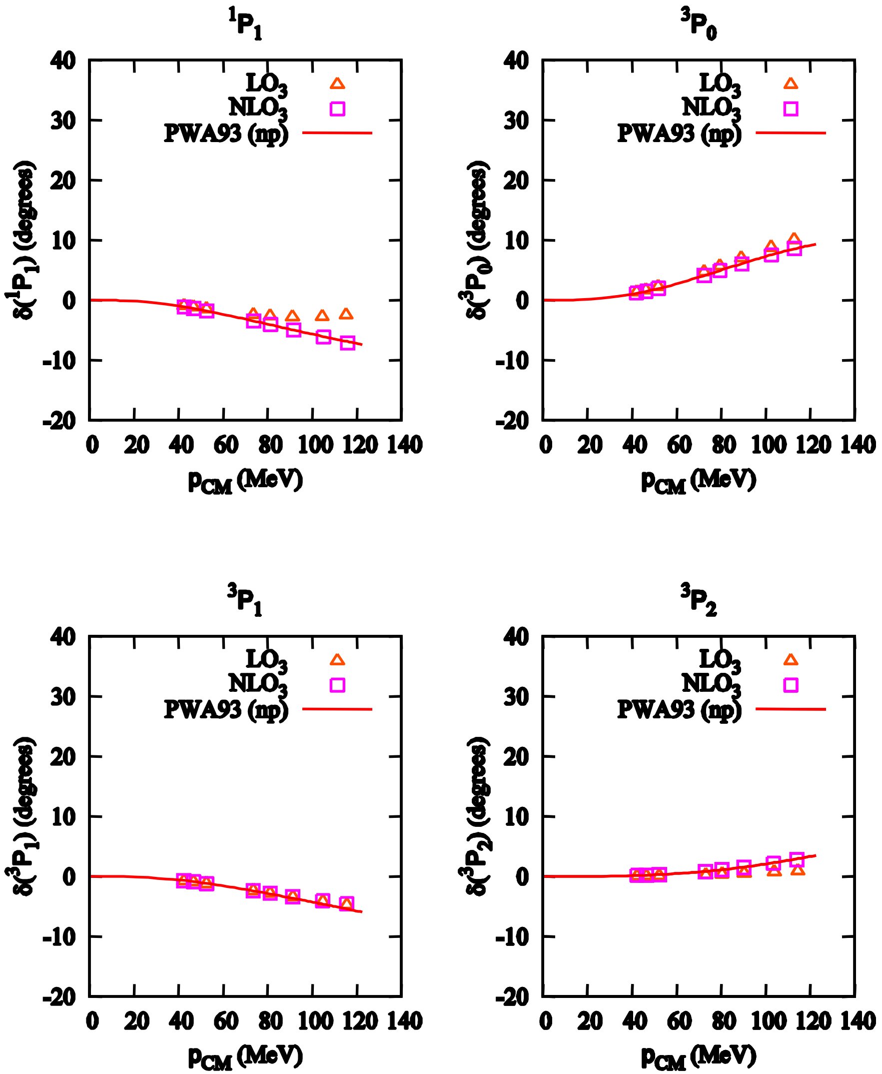
Representation	J_z	Example
A_1	$0 \bmod 4$	$Y_{0,0}$
T_1	$0, 1, 3 \bmod 4$	$\{Y_{1,0}, Y_{1,1}, Y_{1,-1}\}$
E	$0, 2 \bmod 4$	$\left\{Y_{2,0}, \frac{Y_{2,-2}+Y_{2,2}}{\sqrt{2}}\right\}$
T_2	$1, 2, 3 \bmod 4$	$\left\{Y_{2,1}, \frac{Y_{2,-2}-Y_{2,2}}{\sqrt{2}}, Y_{2,-1}\right\}$
A_2	$2 \bmod 4$	$\frac{Y_{3,2}-Y_{3,-2}}{\sqrt{2}}$



LO$_3$: S waves $a = 1.97$ fm



LO$_3$: P waves $a = 1.97$ fm



Dilute neutron matter at NLO

N = 8, 12, 16 neutrons at $L^3 = 4^3, 5^3, 6^3, 7^3$

$a = 1.97$ fm

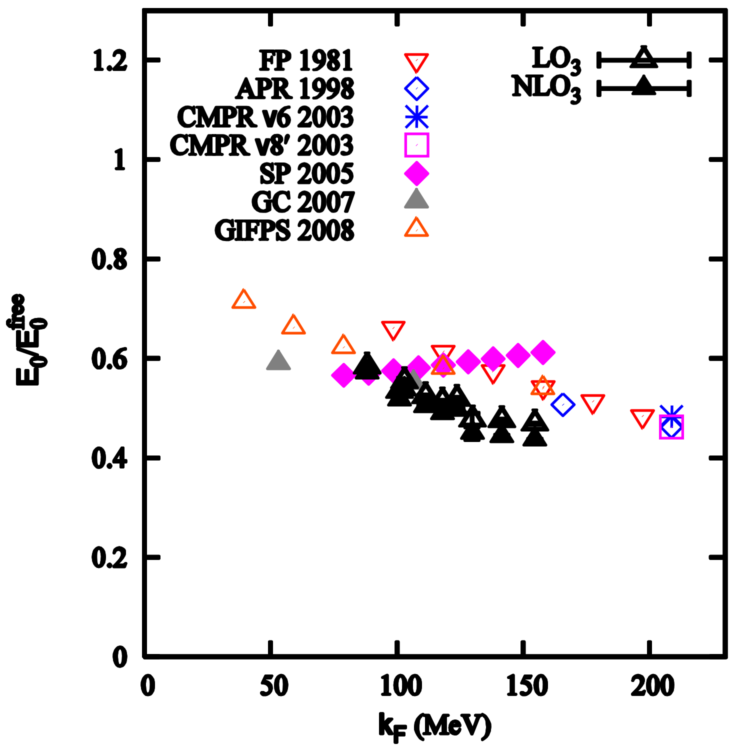
Close to unitarity limit result with scattering length, effective range corrections, etc.

$$\frac{E_0}{E_0^{\text{free}}} = \xi - \frac{\xi_1}{k_F a} + c k_F r_0 + \ldots$$

$$\xi = 0.31(1)$$

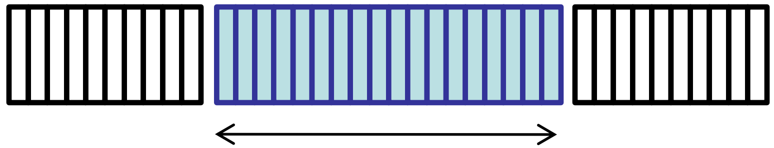
$$\xi_1 \approx 0.8$$

$$c \approx 0.16$$



Epelbaum, Krebs, D.L, Meißner, 0812.3653 [nucl-th]

How bad is the sign problem really?

$\langle \psi_{\text{init}} |$  $| \psi_{\text{init}} \rangle$

$$n_t \sim E_F^{-1}$$

Cost of generating new configurations

$$\sim c_1\, N_{\text{neutrons}} L^3 n_t + c_2\, N_{\text{neutrons}}^3 + \cdots$$

Sign oscillation scaling by far most significant

$$\langle \text{sign} \rangle \sim 2^{-(k_F N_{\text{neutrons}})/(400\,\text{MeV})}$$

Current (0.05 Teraflop years) $k_F N_{\mathrm{neutrons}} = 2000\,\mathrm{MeV}$

6% normal nuclear matter density for 16 neutrons

$$a = 1.97\,\mathrm{fm}$$

Near future (1 Teraflop years) $k_F N_{\mathrm{neutrons}} = 2800\,\mathrm{MeV}$

15% normal nuclear matter density for 16 neutrons

$$a = 1.6\,\mathrm{fm}$$

Petascale (100 Teraflop years) $k_F N_{\mathrm{neutrons}} = 4200\,\mathrm{MeV}$

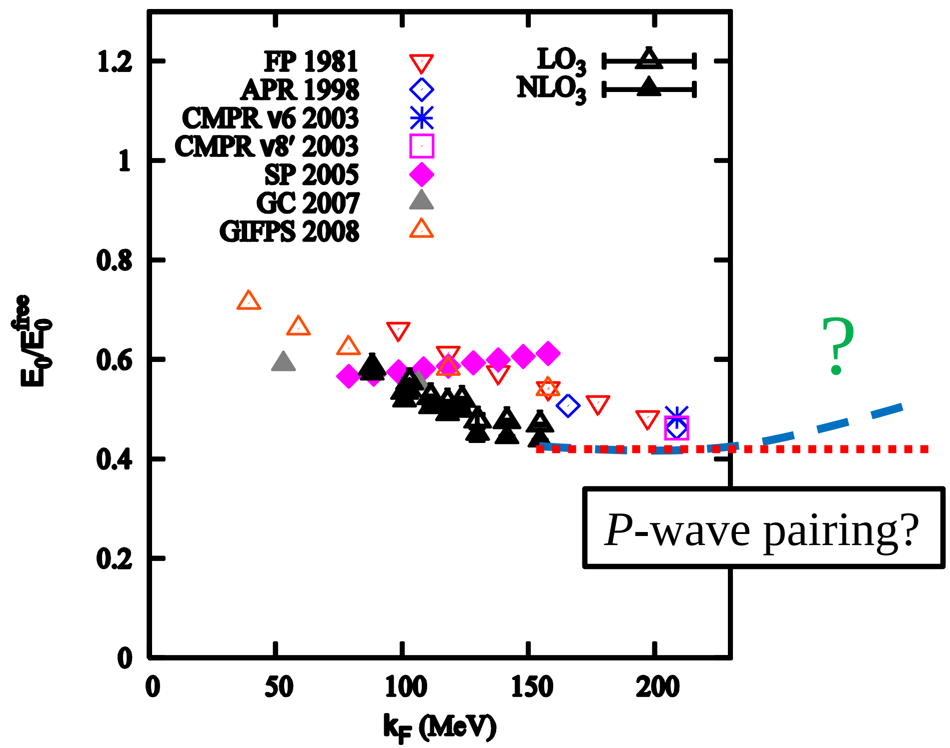
15% normal nuclear matter density for 24 neutrons

$$a = 1.6\,\mathrm{fm}$$

Exascale (100 Petaflop years) $k_F N_{\mathrm{neutrons}} = 6100\,\mathrm{MeV}$

50% normal nuclear matter density for 24 neutrons

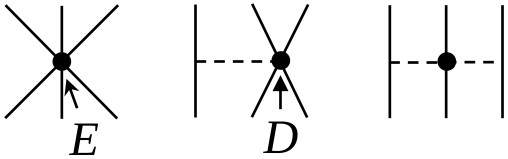
$$a = 1.2\,\mathrm{fm}$$



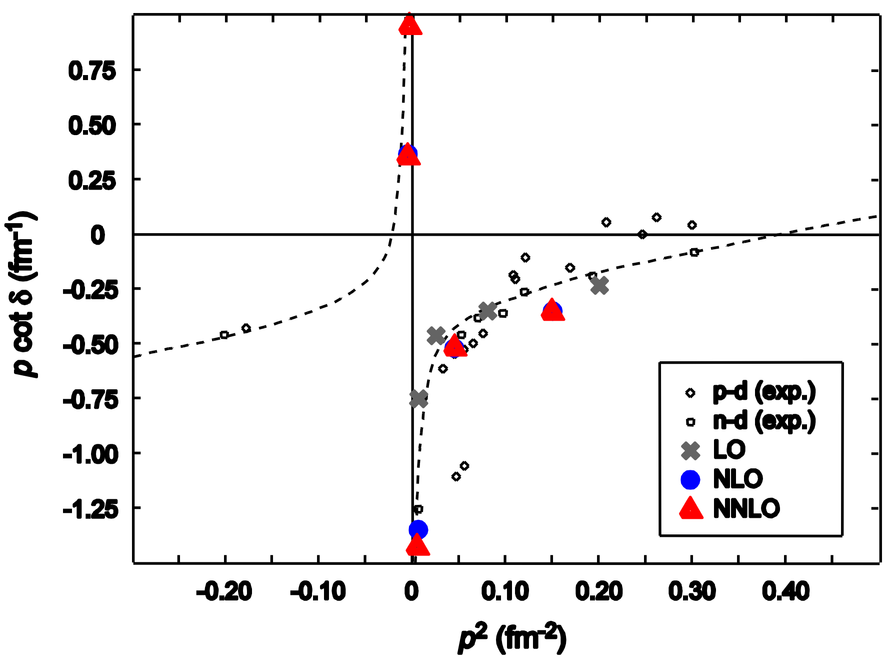
FP 1981
APR 1998
CMPR v6 2003
CMPR v8′ 2003
SP 2005
GC 2007
GIFPS 2008
LO3
NLO3
?
P-wave pairing?
1.2
1
0.8
0.6
0.4
0.2
0
50
100
150
200
kF (MeV)
E0/E0free

Three-body forces at NNLO

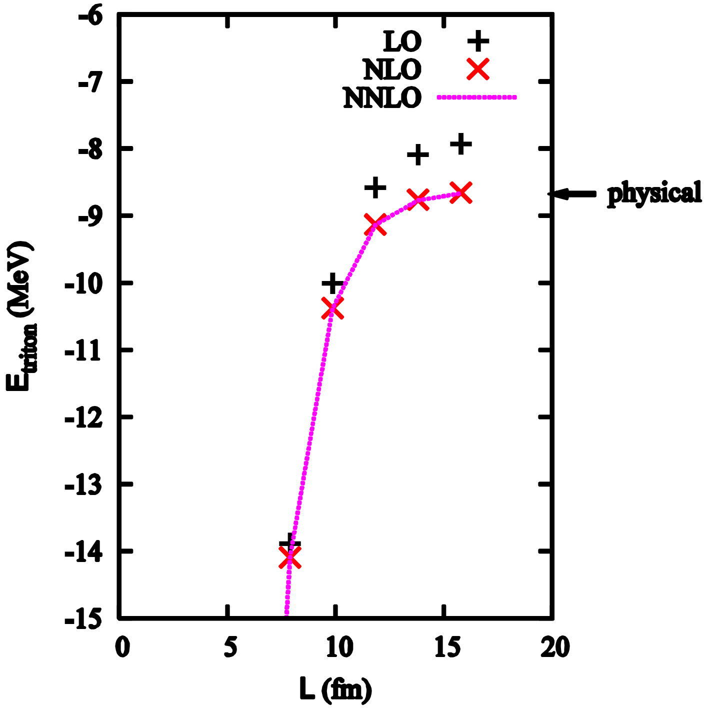
Fit c_D and c_E to spin-1/2 nucleon-deuteron scattering and ^{3}H binding energy



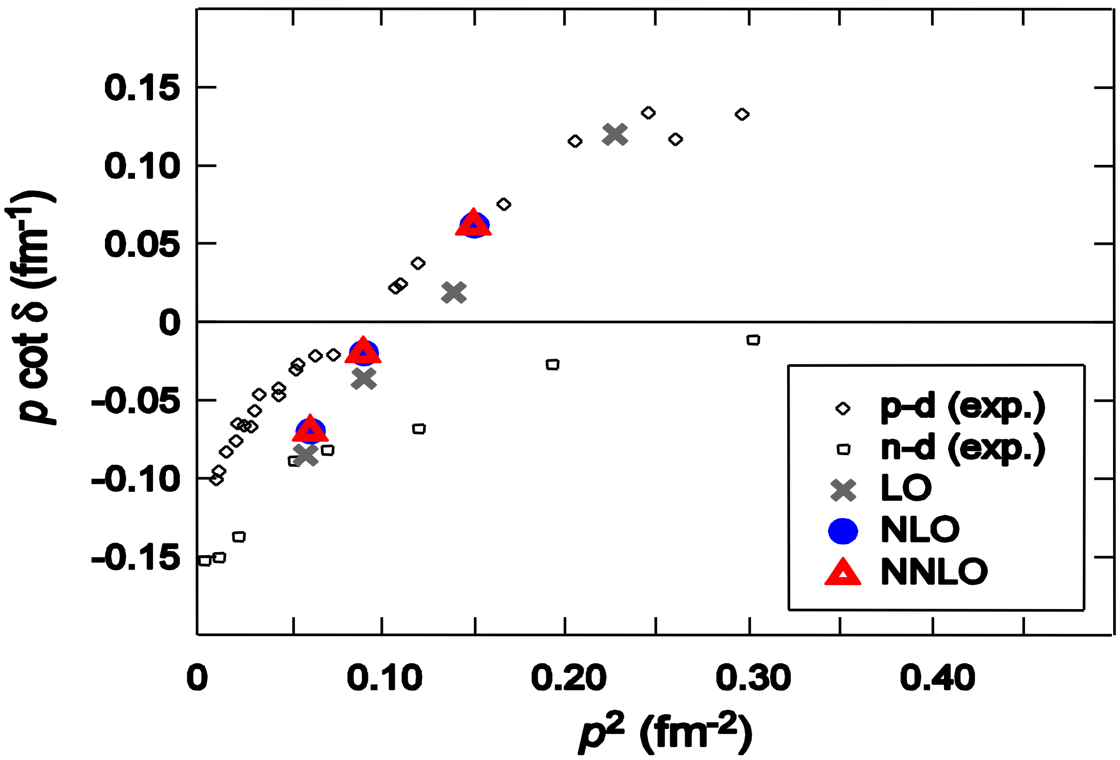
Spin-1/2 nucleon-deuteron



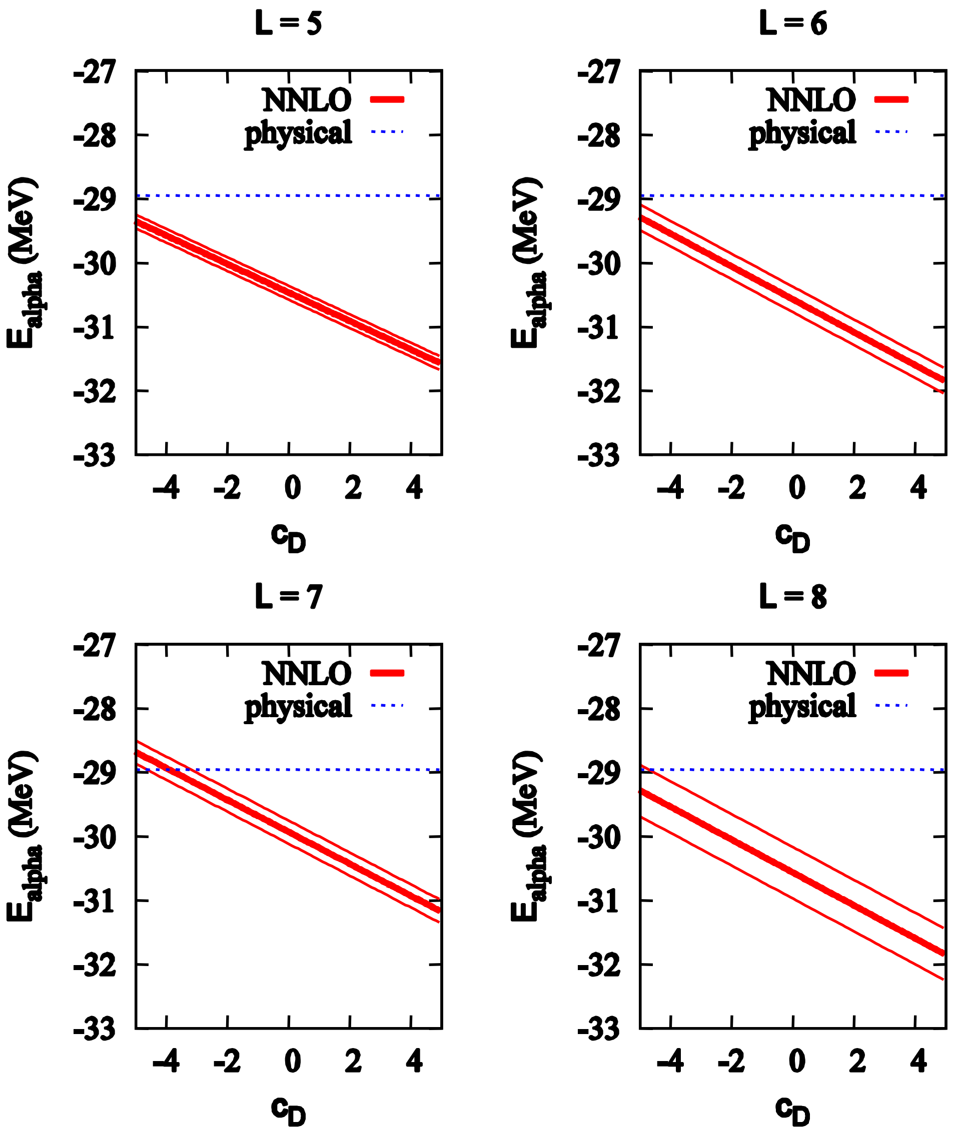
^{3}H binding energy



Spin-3/2 nucleon-deuteron scattering



Alpha-particle energy



Summary

- Promising but relatively new tool that combines the framework of effective field theory and computational lattice methods
- Potentially wide applications to zero and nonzero temperature simulations of cold atoms, light nuclei, neutron matter

Future directions

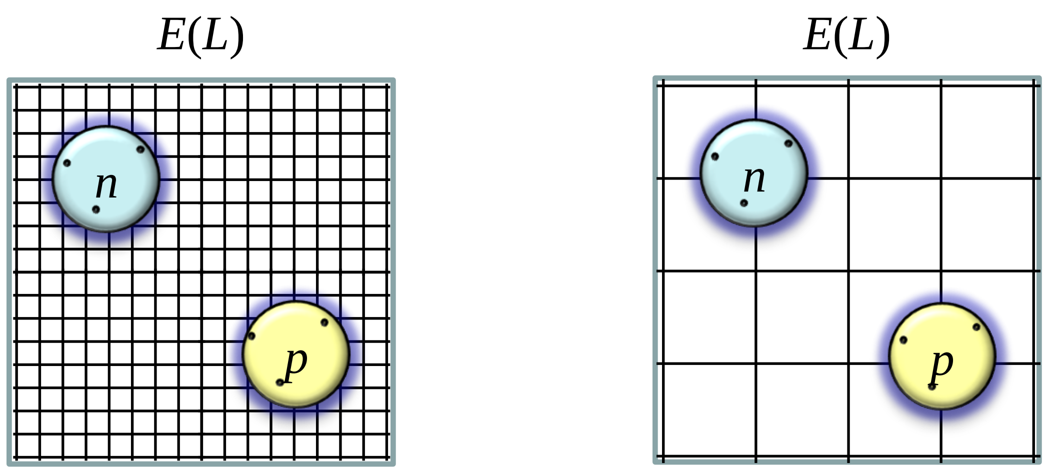
- Improve accuracy – higher order, smaller lattice spacing, larger volume, more nucleons
- Include Coulomb effects and isospin breaking
- Compute nucleon-nucleus scattering, nucleus-nucleus scattering

Connecting lattice QCD and lattice EFT

Finite volume matching for two-nucleon states

For the same periodic volume, compute two-nucleon energies in Lattice QCD and match to two-nucleon energies Lattice EFT

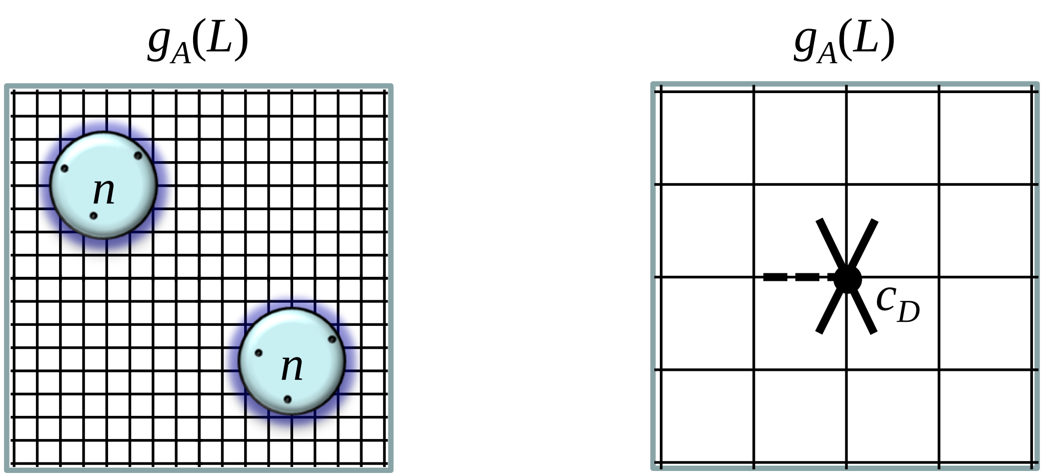
Pion mass dependence?



Connecting lattice QCD and lattice EFT

Calculate g_A for the two-neutron state at finite volume

For given lattice spacing in lattice EFT, use the value of g_A obtained via Lattice QCD at the same volume to fix c_D



Connecting lattice QCD and lattice EFT