Langevin equation and complex actions: are we missing anything?

An introduction to panel discussion

F. Di Renzo

Università di Parma and *INFN, Parma, Italy*

What is this intended for?

I was thinking of a quite informal contribution to the workshop: more (what seem to me) open questions than (tentative) answers.

In the end it was quite natural to turn them into an introduction to a panel discussion!

Nevertheless, apologies in advance to everyone who can't find in what follows the more fundamental and stimulating questions to ask …

… even though it will be up to all of you to proceed to the real thing (the discussion itself!)

Why does Langevin work?

The standard argument: Langevin equation is not a Monte Carlo, it is a stochastic differential equation with a dynamic of its own.

$$\partial_\tau \phi_\eta(x,\tau) = -\frac{\delta S[\phi]}{\delta\phi(x)} + \eta(x,\tau)$$

Still, the standard steps one takes to prove convergence of results to a desired distribution goes through the Fokker-Plank equation

$$\int dP[\eta] O[\phi_\eta(\tau)] = \int D\phi \, O[\phi] P[\phi,\tau]$$

$$\partial_\tau P[\phi,\tau] = \int dx \, \frac{\delta}{\delta\phi(x)} \left(\frac{\delta S[\phi]}{\delta\phi(x)} + \frac{\delta}{\delta\phi(x)} \right) P[\phi,\tau]$$

whose asymptotic solution is usually looked for via an hamiltonian formalism

$$\mathrm{P}(\mathrm{x},\tau) = e^{-S/2}\psi(x,t) \qquad \partial_\tau \psi = -2H_{FP}\psi$$

$$\mathrm{H}_{FP} = \frac{1}{2}\left(-\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial S}{\partial x}\right)\left(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial S}{\partial x}\right) \quad \rightarrow \quad \lim_{\tau\rightarrow\infty} P(x,\tau) \propto e^{-S}$$

What about COMPLEX S?

This is again a standard argument (Parisi, Ambjorn): one starts with a real field, but from Langevin equation one gets a complex solution

$$\begin{aligned}\partial_\tau x(\tau) &= -\Re(\frac{\partial S}{\partial x})|_{x=z} + \eta(\tau) \\ \partial_\tau y(\tau) &= -\Im(\frac{\partial S}{\partial x})|_{x=z}\end{aligned}$$

In the end, this is formally simple: it is just the case of real interacting fields, for which a Fokker-Plank equation can be formulated with much the same (probabilistic) interpretation of the standard case

$$\partial_\tau P(x,y,\tau) = \frac{\partial}{\partial x}(\frac{\partial}{\partial x} + \Re(\frac{\partial S}{\partial x})|_{x=z})P(x,y,\tau) + \frac{\partial}{\partial y}(\Im(\frac{\partial S}{\partial x})|_{x=z}P(x,y,\tau))$$

One would like to rephrase this in terms of the original form one is interested in

$$\int dP[\eta]\, f(z,\tau) = \int dxdy\, f(x+iy)P(x,y,\tau) = \int dx\, f(x)P(x,\tau)$$

Unfortunately a complex Fokker-Plank formalism is not trivial (still, via moments…)

What about SU(3)?

In the case of SU(3) a complex action results in a Langevin process taking place in (another manifold:) SL(3,C).

$$U_{x\mu}(\tau+\epsilon;\eta) = e^{iT^a(\epsilon\nabla^a_{x\mu}S+\sqrt{\epsilon}\eta^a_{x\mu})}\,U_{x\mu}(\tau;\eta)$$

IN GENERAL: one can rephrase in terms of RESTORING FORCES (fixed points, attractors, …) With that respect, it is clear there could be a nightmare situation: an unbounded diffusion in the "out of the original manifold" degrees of freedom!

- ✓ Any hope to tackle REAL probability distributions?
- ✓ What kind of results/evidence on BOUNDED DIFFUSION out of SU(3)?

A KEY ISSUE: what is the interplay of this diffusion with the diffusion in the gauge orbits?

Stochastic gauge fixing revisited?

A funny story: all the business started as "Perturbation Theory without gauge fixing" (Parisi-Wu), but … beware (QED suffices to understand …)

$$\begin{aligned} \partial_\tau \, T_{\mu\nu}(k) A_\nu(k,\tau) &= -k^2 \, T_{\mu\nu}(k) A_\nu(k,\tau) + T_{\mu\nu}(k)\, \eta_\nu(k,\tau) \\ \partial_\tau \, L_{\mu\nu}(k) A_\nu(k,\tau) &= L_{\mu\nu}(k)\, \eta_\nu(k,\tau) \end{aligned}$$

i.e. NO RESTORING FORCE for longitudinal degrees of freedom! Notice that:

- Gauge invariants quantities are unaffected by these diverences. (Beware: numerical cancelations can be bad)
- Still, the standard approach to demonstrate the approach to the desired asymptotic distribution fails (and it fails at free fields level! It does not come as a surprise: this is a close relative to the failure in computing the tree level Feynman propagator in standard PT)

The way out: add an extra piece!

$$\partial_\tau A^a_\mu(x;\tau) = -\frac{\delta S[A]}{\delta A^a_\mu(x;\tau)} - D^{ab}_\mu V^b[A,\tau] + \eta^a_\mu(x;\tau)$$

Gauge invariant quantities are unaffected

$$\partial_\tau F[A] = \int dx \, \frac{\delta F[A]}{\delta A_\mu^a(x;t)} \frac{\partial A_\mu^a(x;t)}{\partial t} = 0$$

But now a free field probability distribution can be written down and a series expansion for the interacting asymptotic distribution can be formulated (which is the desired one).

I think stochastic gauge fixing has been reported as valuable in the real-time Langevin simulations framework (Isn't it?)

- ✓ Has it been tested for finite μ?
- ✓ Could it be that some generalization of it can be useful in this framework?

I made no reference to a recently appeared paper: "Effective Potential for Complex Langevin Equations", G. Guralnik, C. Pehlevan, arXiv 0902.1503.

- ✓ Has anybody carefully read it?

Well, these were only naive (informal) remarks rephrased as an introduction to panel discussion …

Now the real thing comes in! … which is up to ALL OF YOU!