Random matrix theory as playground for the sign problem

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Outline

1. Lattice QCD and chiral random matrix theory with chemical potential
2. Phase of fermion determinant in chRMT
3. Quenched and unquenched chRMT simulations

Chiral symmetry and quark chemical potential

- QCD at non-zero quark density → introduce quark chemical potential μ
- Continuum fermion action at with chemical potential

$$S_F = \int d^4x \, \bar{\psi}(x)(m + \gamma_\mu \partial^\mu_{\text{cov}} + \mu\gamma_4)\psi(x)$$

- massless quarks: action invariant under independent transformation of left-handed and right-handed fermions → chiral symmetry

$$\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$$

- $U(1)_A$ anomaly + dynamical chiral symmetry breaking

Chiral symmetry and chemical potential on the lattice

- Chiral symmetry on the lattice $\longrightarrow$ Ginsparg-Wilson relation
- Satisfied by overlap Dirac operator of Neuberger-Narayanan
- Generalization to $\mu \neq 0$ (JB, Wettig PRL97(012003) 2006):

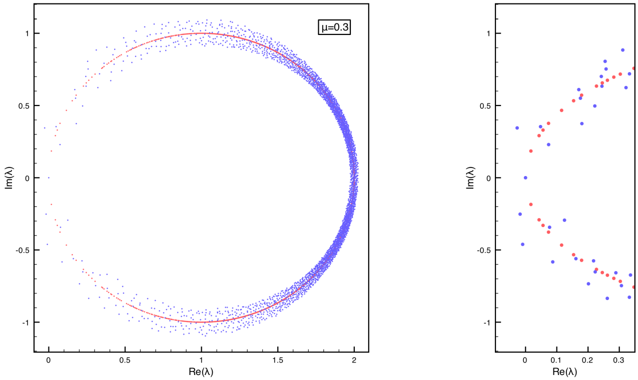
$$D_{\text{ov}}(\mu) = \mathbb{1} + \gamma_5 \,\text{sgn}(\gamma_5 D_W(\mu))$$

where $D_W(\mu)$ is Wilson-Dirac operator at $\mu \neq 0$: (Hasenfratz-Karsch 1983, Kogut et al. 1983)

$$D_W(\mu) = 1 - \kappa \sum_{i=1}^{3} (T_i^+ + T_i^-) - \kappa (e^{\mu} T_4^+ + e^{-\mu} T_4^-)$$

with $\quad (T_\nu^\pm)_{yx} = (1 \pm \gamma_\nu) U_{x,\pm\nu} \delta_{y,x\pm\hat{\nu}}$

Typical spectrum ($V = 4^4, \beta = 5.1, m_W = -2, \mu = 0.3$)



- $D_{\text{ov}}(\mu)$ satisfies Ginsparg-Wilson relation → exact zero modes with definite chirality reflecting topological charge of gauge configuration.
- naturally violates γ_5-Hermiticity → spectrum no longer on circle
- Sign problem at $\mu \neq 0$: $\det[D_{\text{ov}}(\mu)]$ is complex → can no longer be incorporated in probability distribution for MCMC

Chiral random matrix model

- Leading order in ε-regime of QCD: spectral properties of Dirac operator are universal and can be described by chiral random matrix theory
- Two-matrix model (Osborn):

$$D(\mu) = \begin{pmatrix} 0 & i\phi_1 + \mu\phi_2 \\ i\phi_1^\dagger + \mu\phi_2^\dagger & 0 \end{pmatrix}$$

where ϕ_1 and ϕ_2 are complex $(N+\nu)\times N$ matrices distributed as

$$w(X) = \exp(-N \operatorname{tr} X^\dagger X)$$

- ν zero modes, non-zero modes come in $(z, -z)$-pairs

Partition function

- After diagonalization the partition function is:

$$Z_\nu(\alpha, m) = \int_{\mathbb{C}} \prod_{k=1}^{N} d^2 z_k \, w^\nu(z_k, z_k^*; \alpha) \, |\Delta_N(\{z^2\})|^2$$

with weight function ($\alpha = \mu^2$):

$$w^\nu(z, z^*; \alpha) = |z|^{2\nu+2} \exp\left[-\tfrac{N(1-\alpha)}{4\alpha}(z^2 + z^{*2}) \right] K_\nu \left[\tfrac{N(1+\alpha)}{2\alpha} |z|^2 \right]$$

and Vandermonde determinant: $\Delta_N(\{z^2\}) \equiv \prod_{i>j=1}^{N} (z_i^2 - z_j^2)$

- Introduce orthogonal polynomials wrt $w^\nu(z, z^*; \alpha)$:

$$p_k^\nu(z; \alpha) = \left(\frac{1-\alpha}{N} \right)^k k! \, L_k^\nu \left[-\tfrac{N z^2}{1-\alpha} \right]$$

where $L_k^\nu(z)$ are generalized Laguerre polynomials.

Microscopic spectral density: LQCD versus chRMT

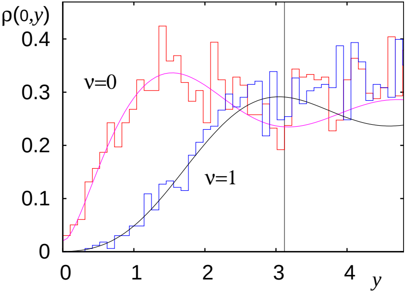
- chRMT: quenched microscopic spectral density

$$\rho_\nu(z) = \frac{|z|^2}{2\pi\alpha} e^{-\frac{z^2+z^{*2}}{8\alpha}} K_\nu\left(\frac{|z|^2}{4\alpha}\right) \int_0^1 dt\, t e^{-2\alpha t^2} |I_\nu(tz)|^2$$

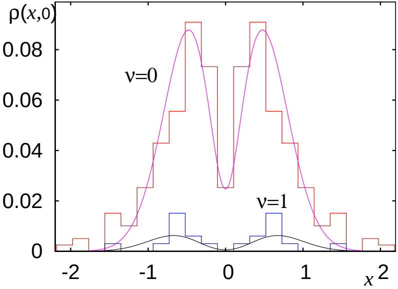
Akemann, Osborn, Splittorff, Verbaarschot 2004-2005

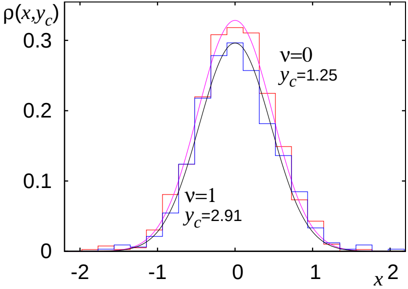
- Correspondence chRMT $\longleftrightarrow$ LQCD: $z = \lambda V \Sigma$ and $\alpha = \mu^2 f_\pi^2 V$
- study spectral distribution for small eigenvalues
  - consider cuts parallel to imaginary and real axes
- Σ and f_π can be extracted by fitting LQCD data to chRMT
  $\longrightarrow$ good agreement for small eigenvalues
- LQCD: overlap fermions have exact zero modes $\longrightarrow$ test RMT predictions for non-trivial topology($\nu \neq 0$)

Spectral density, $V = 4^4$, $\mu = 0.1$, 8703 config.



Σa^3	$f_\pi a$	χ^2/dof
0.0812(11)	0.261(6)	0.67





Phase factor of fermion determinant in chRMT

- massive Dirac operator in QCD: $D(m;\mu) = m\mathbb{1} + D(\mu)$
- average phase factor: characterizes oscillatory behavior of fermion determinant connected to sign problem in dynamical simulations
- set $\det[D(m;\mu)] \equiv Re^{i\theta}$. The phase factor of the squared determinant is:

$$e^{2i\theta} = \frac{\det(D(\mu)+m)}{\det(D^\dagger(\mu)+m)} = \prod_{k=1}^{N} \frac{m^2 - z_k^2}{m^2 - z_k^{*2}}$$

- Unquenched ensemble average:

$$\left\langle e^{2i\theta} \right\rangle_{N_f} = \left\langle \frac{\det(D(\mu)+m)}{\det(D^\dagger(\mu)+m)} \right\rangle_{N_f} = \frac{Z_\nu^{N_f+1|1^*}(\alpha, m)}{Z_\nu^{N_f}(\alpha)}$$

$$= \frac{1}{Z_\nu^{N_f}} \int_{\mathbb{C}} \prod_{k=1}^{N} d^2 z_k \, w^\nu(z_k, z_k^*; \alpha) \, |\Delta_N(\{z^2\})|^2 \frac{m^2 - z_k^2}{m^2 - z_k^{*2}} \prod_{f=1}^{N_f} (m_f^2 - z_k^2),$$

- derivation for general topology follows ideas of:
  - Splittorff and Verbaarschot, Phys. Rev. D75 (2007) 116003
  - Osborn, Splittorf and Verbaarschot, Phys. Rev. D78 (2008) 065029.

Phase factor: quenched case

From Akemann, Pottier and Bergère (2004):

$$\left\langle e^{2i\theta}\right\rangle_{N_f=0} = \begin{vmatrix} \mathscr{H}_{\nu,0}(\alpha,m) & \mathscr{H}_{\nu,1}(\alpha,m) \\ p_{N-1}^{\nu,0}(m;\alpha) & p_{N-1}^{\nu,1}(m;\alpha) \end{vmatrix}$$

with the complex Cauchy transform:

$$\mathscr{H}_{\nu,k}(\alpha,m) = -\frac{1}{r_{N-1}^{\nu}(\alpha)} \int_{\mathbb{C}} \frac{d^2 z}{z^2 - m^2} w^{\nu}(z,z^*;\alpha) p_{N-1}^{\nu,k}(z^*)\,,$$

where we defined:

$$p_{\ell}^{\nu,k}(z;\alpha) = z^{2k} p_{\ell}^{\nu+k}(z;\alpha)\,.$$

and r_{N-1}^{ν} is weighted norm of orthogonal polynomial p_{N-1}^{ν}

$$r_k^{\nu}(\alpha) = \frac{\pi\alpha(1+\alpha)^{2k+\nu} k!(k+\nu)!}{N^{2k+\nu+2}}\,.$$

- Integrand of $\mathscr{H}_{\nu,k}(\alpha,m)$ strongly oscillates in $\mathrm{Im}\, z$ direction

Phase factor: unquenched case

Again using Akemann and Pottier:

$$\langle e^{2i\theta}\rangle_{N_f} = \frac{1}{\prod_{f=1}^{N_f}(m_f^2 - m^2)} \frac{\begin{vmatrix} \mathcal{H}_{\nu,0}(\alpha,m) & \mathcal{H}_{\nu,1}(\alpha,m) & \cdots & \mathcal{H}_{\nu,N_f+1}(\alpha,m) \\ p_{N-1}^{\nu,0}(m;\alpha) & p_{N-1}^{\nu,1}(m;\alpha) & \cdots & p_{N-1}^{\nu,N_f+1}(m;\alpha) \\ p_{N-1}^{\nu,0}(m_1;\alpha) & p_{N-1}^{\nu,1}(m_1;\alpha) & \cdots & p_{N-1}^{\nu,N_f+1}(m_1;\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ p_{N-1}^{\nu,0}(m_{N_f};\alpha) & p_{N-1}^{\nu,1}(m_{N_f};\alpha) & \cdots & p_{N-1}^{\nu,N_f+1}(m_{N_f};\alpha) \end{vmatrix}}{\begin{vmatrix} p_N^{\nu,0}(m_1;\alpha) & p_N^{\nu,1}(m_1;\alpha) & \cdots & p_N^{\nu,N_f-1}(m_1;\alpha) \\ p_N^{\nu,0}(m_2;\alpha) & p_N^{\nu,1}(m_2;\alpha) & \cdots & p_N^{\nu,N_f-1}(m_2;\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ p_N^{\nu,0}(m_{N_f};\alpha) & p_N^{\nu,1}(m_{N_f};\alpha) & \cdots & p_N^{\nu,N_f-1}(m_{N_f};\alpha) \end{vmatrix}}$$

Equal mass quarks

Take limit that valence quark and all dynamical quarks have same mass m.
Taylor expansion around m:

$$p_\ell^{\nu,k}(m_f;\alpha) = p_\ell^{\nu,k}(m;\alpha) + \sum_{j=1}^{\infty} \frac{(m_f - m)^j}{j!}\, \partial_m^j p_\ell^{\nu,k}(m;\alpha)$$

Phase factor with N_f equal mass dynamical fermions:

$$\left\langle e^{2i\theta} \right\rangle_{N_f} = \frac{1}{(2m)^{N_f} N_f!} \frac{\begin{vmatrix} \mathcal{H}_{\nu,0}(\alpha,m) & \mathcal{H}_{\nu,1}(\alpha,m) & \cdots & \mathcal{H}_{\nu,N_f+1}(\alpha,m) \\ p_{N-1}^{\nu,0}(m;\alpha) & p_{N-1}^{\nu,1}(m;\alpha) & \cdots & p_{N-1}^{\nu,N_f+1}(m;\alpha) \\ \partial_m p_{N-1}^{\nu,0}(m;\alpha) & \partial_m p_{N-1}^{\nu,1}(m;\alpha) & \cdots & \partial_m p_{N-1}^{\nu,N_f+1}(m;\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_m^{N_f} p_{N-1}^{\nu,0}(m;\alpha) & \partial_m^{N_f} p_{N-1}^{\nu,1}(m;\alpha) & \cdots & \partial_m^{N_f} p_{N-1}^{\nu,N_f+1}(m;\alpha) \end{vmatrix}}{\begin{vmatrix} p_N^{\nu,0}(m;\alpha) & p_N^{\nu,1}(m;\alpha) & \cdots & p_N^{\nu,N_f-1}(m;\alpha) \\ \partial_m p_N^{\nu,0}(m;\alpha) & \partial_m p_N^{\nu,1}(m;\alpha) & \cdots & \partial_m p_N^{\nu,N_f-1}(m;\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_m^{N_f-1} p_N^{\nu,0}(m;\alpha) & \partial_m^{N_f-1} p_N^{\nu,1}(m;\alpha) & \cdots & \partial_m^{N_f-1} p_N^{\nu,N_f-1}(m;\alpha) \end{vmatrix}}.$$

Microscopic limit

- Take $N \to \infty$, keep $\hat{\alpha} = 2N\alpha$, $\hat{m} = 2Nm$ fixed
- Phase factor with N_f equal mass dynamical fermions:

$$\langle e_s^{2i\theta} \rangle_{N_f} = \frac{1}{(2\hat{m})^{N_f} N_f!} \frac{\begin{vmatrix} \mathcal{H}^s_{\nu,0}(\hat{\alpha},\hat{m}) & \mathcal{H}^s_{\nu,1}(\hat{\alpha},\hat{m}) & \cdots & \mathcal{H}^s_{\nu,N_f+1}(\hat{\alpha},\hat{m}) \\ I_{\nu,0}(\hat{m}) & I_{\nu,1}(\hat{m}) & \cdots & I_{\nu,N_f+1}(\hat{m}) \\ I'_{\nu,0}(\hat{m}) & I'_{\nu,1}(\hat{m}) & \cdots & I'_{\nu,N_f+1}(\hat{m}) \\ \vdots & \vdots & \vdots & \vdots \\ I^{(N_f)}_{\nu,0}(\hat{m}) & I^{(N_f)}_{\nu,1}(\hat{m}) & \cdots & I^{(N_f)}_{\nu,N_f+1}(\hat{m}) \end{vmatrix}}{\begin{vmatrix} I_{\nu,0}(\hat{m}) & I_{\nu,1}(\hat{m}) & \cdots & I_{\nu,N_f-1}(\hat{m}) \\ I'_{\nu,0}(\hat{m}) & I'_{\nu,1}(\hat{m}) & \cdots & I'_{\nu,N_f-1}(\hat{m}) \\ \vdots & \vdots & \vdots & \vdots \\ I^{(N_f-1)}_{\nu,0}(\hat{m}) & I^{(N_f-1)}_{\nu,1}(\hat{m}) & \cdots & I^{(N_f-1)}_{\nu,N_f-1}(\hat{m}) \end{vmatrix}}$$

with $I_{\nu,k}(z) = z^k I_{\nu+k}(z)$ and

$$\mathcal{H}^s_{\nu,k}(\hat{\alpha},\hat{m}) = -\frac{e^{-2\hat{\alpha}}}{4\pi\hat{\alpha}\hat{m}^\nu} \int_{\mathbb{C}} \frac{d^2z}{z^2-\hat{m}^2} \frac{|z|^{2(\nu+1)}}{z^{*\nu}} \exp\left(-\frac{z^2+z^{*2}}{8\hat{\alpha}}\right) K_\nu\left(\frac{|z|^2}{4\hat{\alpha}}\right) I_{\nu,k}(z^*)$$

Solving the complex Cauchy transform – microscopic limit

Consider the complex integral:

$$\mathcal{H}^{s}_{\nu,k}(\hat{\alpha},\hat{m}) = -\frac{e^{-2\hat{\alpha}}}{4\pi\hat{\alpha}\hat{m}^{\nu}} \int_{\mathbb{C}} \frac{d^2z}{z^2-\hat{m}^2} \frac{|z|^{2(\nu+1)}}{z^{*\nu}} \exp\left(-\tfrac{z^2+z^{*2}}{8\hat{\alpha}}\right) K_{\nu}\left(\tfrac{|z|^2}{4\hat{\alpha}}\right) I_{\nu,k}(z^*)$$

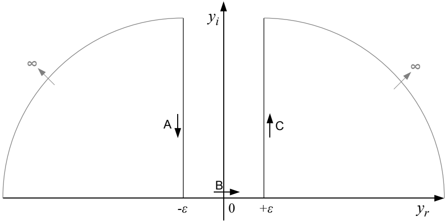
- asymptotic behavior of kernel ($z = x + iy$):
  - x-direction: Gaussian with width $\sqrt{\hat{\alpha}}$
  - y-direction: rapid oscillations inside an envelope that diverges like $y^{\nu+k-3/2}$
  - integral represents observable quantity in RMT $\longrightarrow$ has to be finite
  - requires particular cancellations due to the oscillatory behavior of integrand
- rewrite:

$$I_{\nu}(z) = \frac{i\eta(z)}{\pi}\left((-)^{\nu}K_{\nu}(z) - K_{\nu}(-z)\right)$$

with $\eta(z) = \pm 1$ for $\mathrm{Im}\, z \gtrless 0$.

Deforming the integration path

- continue $y \to y_r + iy_i$ in $z = x + iy$
- deform the y-integration into the complex y-plane to U-shaped contour:



- $K_\nu(\pm z^*)$: close contour in $y_i \gtrless 0$ plane
- Introduce changes of variables $y = -is \pm \varepsilon$, and $t = x - s$, $u = x + s$

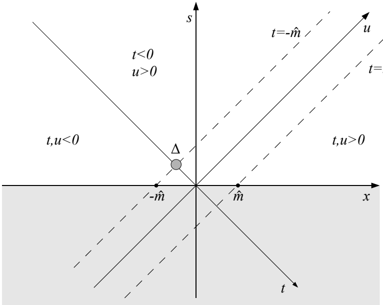
$$\mathscr{H}^s_{\nu,k}(\hat{\alpha},\hat{m}) = \lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} dt \int_t^{\infty} du \left[f_+(t,u) + f_-(t,u)\right],$$

with

$$f(t,u) = \frac{e^{-2\hat{\alpha}}}{4\pi^2\hat{\alpha}\hat{m}^\nu} \frac{(-)^{\nu+k+1} t^{\nu+1} u^{k+1}}{t^2 - \hat{m}^2} \exp\left[-\frac{t^2+u^2}{8\hat{\alpha}}\right] K_\nu\left[\frac{tu}{4\hat{\alpha}}\right] K_{\nu+k}[u]$$

Integration plane

- Integration plane:



- Integrate over two semi-infinite sheets $S_{\pm} \parallel (t,u)$-plane
- In the (t,u)-plane integrand has:
  - two mass-pole lines at $t = \pm\hat{m}$
  - branch cuts of Bessel-K function for negative real argument
  - singularity of Bessel-K function for zero argument: $K_{\nu}(z) \overset{z\to 0}{\sim} z^{-\nu}$

- Apply the Sokhatsky-Weierstrass theorem

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{f(x)}{x\pm i\varepsilon}dx = \mp i\pi f(0) + \mathrm{PV}\int_a^b \frac{f(x)}{x}dx$$

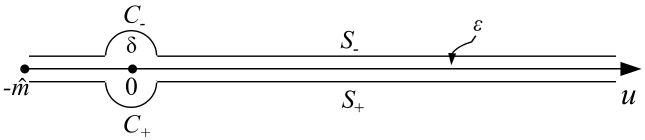
on the t-integral of $S_\pm$, where PV denotes the Cauchy principal value integral.

- Principal value integral vanishes because of symmetry considerations
- When $S_\pm$ converge to (t,u)-plane $\rightarrow$ K-Bessel has branch cut discontinuity on negative real axis:

$$\lim_{\varepsilon\to 0^+} K_\nu(x\pm i\varepsilon) = \begin{cases} K_\nu(|x|) & \text{if} \quad x>0, \\ (-)^\nu K_\nu(|x|) \mp i\pi I_\nu(|x|) & \text{if} \quad x<0. \end{cases}$$

So, distinguish 3 integration regions:

$$\lim_{\varepsilon\to 0^+}\Big[K_\nu(tu)K_{\nu+j}(u)\Big]_\pm = \begin{cases} t,u>0 & \text{no b.c.,} \\ t<0<u & \text{b.c. in } K(tu), \\ t,u<0 & \text{b.c. in } K(u). \end{cases}$$



- Residue terms for $t = \pm\hat{m}$:
  - branch cut discontinuity and $u = 0$-singularity contribute
    $\rightarrow$ $t = \hat{m}$ contribution vanishes as $t, u > 0$, only $t = -\hat{m}$ contributes
  - deform contour to circumvent the $u = 0$ singularity for $\nu \neq 0$

$$\mathcal{H}^{s}_{\nu,k}(\hat{\alpha},\hat{m}) = \frac{e^{-2\hat{\alpha}-\frac{\hat{m}^2}{8\hat{\alpha}}}}{4\hat{\alpha}}\Bigg[\int_0^\infty du(-)^k u^{k+1}\exp\left[-\tfrac{u^2}{8\hat{\alpha}}\right] I_\nu\left[\tfrac{\hat{m}u}{4\hat{\alpha}}\right] K_{\nu+k}[u] + \int_0^{\hat{m}} du\, u^{k+1}\exp\left[-\tfrac{u^2}{8\hat{\alpha}}\right] K_\nu\left[\tfrac{\hat{m}u}{4\hat{\alpha}}\right] I_{\nu+k}[u]\Bigg] + \Delta_{\nu,k}(\hat{\alpha},\hat{m})$$

where

$$\Delta_{\nu,k}(\hat{\alpha},\hat{m}) = \frac{ie^{-2\hat{\alpha}-\frac{\hat{m}^2}{8\hat{\alpha}}}}{8\pi\hat{\alpha}}\oint_{\Gamma_0} du\,(-)^k u^{k+1}\exp\left[-\tfrac{u^2}{8\hat{\alpha}}\right] K_\nu\left[\tfrac{-\hat{m}u}{4\hat{\alpha}}\right] K_{\nu+k}[u]$$

Contribution of $u = 0$ singularity ($\nu \neq 0$)

- Compute contour integral with residue theorem

$$\Delta_{\nu,k}(\hat{\alpha},\hat{m}) = -\frac{e^{-2\hat{\alpha}-\frac{\hat{m}^2}{8\hat{\alpha}}}}{4\hat{\alpha}} \operatorname*{Res}_{u=0}\left[(-)^k u^{k+1} \exp\left[-\frac{u^2}{8\hat{\alpha}}\right] K_\nu\left[\frac{-\hat{m}u}{4\hat{\alpha}}\right] K_{\nu+k}[u]\right]$$

- Residue is given by coefficient of simple pole in Laurent expansion – use small z Expansion of $\exp(z)$ and $K_\nu(z)$:

$$\Delta_{\nu,k}(\hat{\alpha},\hat{m}) = e^{-2\hat{\alpha}-\frac{\hat{m}^2}{8\hat{\alpha}}} \frac{(-)^k 2^{\nu+k-1}}{\hat{m}^\nu} \times \sum_{i=0}^{\nu-1}\sum_{j=0}^{\nu-i-1} \frac{(\nu-1-i)!(\nu+k-1-j)!}{(\nu-1-i-j)!i!j!}(2\hat{\alpha})^j\left(\frac{\hat{m}^2}{8\hat{\alpha}}\right)^i$$

$\Delta_{\nu,k}(\hat{\alpha},\hat{m})$ is polynomial of degree $(\nu-1)$ in $\hat{\alpha}$ and $\hat{m}^2/\hat{\alpha}$

Solving the complex Cauchy transform - finite N

$$\mathcal{H}_{\nu,k}(\alpha,m) = -\frac{1}{r^{\nu}_{N-1}(\alpha)} \int_{\mathbb{C}} \frac{d^2z}{z^2 - m^2} |z|^{2(\nu+1)} e^{-a(z^2+z^{*2})} K_\nu(b|z|^2) p^{\nu,k}_{N-1}(z^*;\alpha)$$

with $a = N(1-\alpha)/4\alpha$ and $b = N(1+\alpha)/2\alpha$. Rewrite à la Osborn et al.

$$\frac{e^{-az^2}}{z^2 - m^2} = \frac{e^{-am^2}}{z^2 - m^2} + \frac{e^{-az^2} - e^{-am^2}}{z^2 - m^2}$$

- Expand first term in geometric series and perform angular integration:

$$\mathcal{H}^{(1)}_{\nu,k}(\alpha,m) = \frac{\pi m^\nu e^{-am^2}}{r^{\nu}_{N-1}(\alpha)} \int_0^m du\, u^{\nu+1} K_\nu(bmu) e^{-au^2} p^{\nu,k}_{N-1}(u;\alpha)$$

- The second term is analytical and can be expanded as

$$\frac{e^{-az^2} - e^{-am^2}}{z^2 - m^2} = e^{-az^2} \sum_{j=0}^{\infty} d_j(m) p^{\nu}_j(z;\alpha)\,,$$

where the m-dependent coefficients $d(m)$ were computed by Osborn et al.

$$d_k(m) = -\frac{N^{k+1}}{k!(1-\alpha)^{k+1}} \int_0^{(1-\alpha)^2/4\alpha} dt\, e^{-m^2Nt/(1-\alpha)} \frac{t^k}{(t+1)^{k+\nu+1}}\,.$$

$$\mathscr{H}^{(2)}_{\nu,k}(\alpha,m) = -\frac{1}{r^{\nu}_{N-1}(\alpha)}\sum_{j=0}^{\infty} d_j(m)\int_{\mathbb{C}} d^2z\, w^{\nu}(z,z^*;\alpha)p^{\nu}_{j}(z;\alpha)p^{\nu,k}_{N-1}(z^*;\alpha)\,.$$

- $k=0$: use orthogonality relation
- $k\neq 0$: first decompose $p^{\nu,k}_{\ell}$ in orthogonal polynomials p^{ν}_{j}

$$p^{\nu,k}_{\ell}(z;\alpha) \equiv z^{2k}p^{\nu+k}_{\ell}(z;\alpha) = \sum_{j=0}^{k}(-)^j\binom{k}{j}\frac{(\ell+\nu+k)!}{(\ell+\nu+k-j)!}\left(\frac{1-\alpha}{N}\right)^j p^{\nu}_{\ell+k-j}(z;\alpha)\,,$$

using recurrence relation for Laguerre polynomials. So,

$$\int_{\mathbb{C}} d^2z\, w^{\nu}(z,z^*;\alpha)p^{\nu}_{\ell}(z;\alpha)p^{\nu,k}_{N-1}(z^*;\alpha)$$
$$= \sum_{j=0}^{k}(-)^j\binom{k}{j}\frac{(N-1+\nu+k)!}{(N-1+\nu+k-j)!}\left(\frac{1-\alpha}{N}\right)^j r^{\nu}_{N-1+k-j}(\alpha)\delta_{\ell,N-1+k-j}\,.$$

$$\mathscr{H}^{(2)}_{\nu,k}(\alpha,m) = \frac{N^{N-k}}{(1-\alpha)^{N+k}}\frac{(N-1+\nu+k)!}{(N-1)!(N-1+\nu)!}$$
$$\times\int_0^{(1-\alpha)^2/4\alpha} dt\, e^{-m^2Nt/(1-\alpha)}\frac{t^{N-1}}{(t+1)^{N+k+\nu}}\left[4\alpha t-(1-\alpha)^2\right]^k$$

Solving the complex Cauchy transform

Finite-N:

$$\mathcal{H}_{\nu,k}(\alpha,m) = \frac{\pi m^{\nu} e^{-am^2}}{r^{\nu}_{N-1}(\alpha)} \int_0^m du\, u^{\nu+1} K_{\nu}(bmu) e^{-au^2} p^{\nu,k}_{N-1}(u;\alpha)$$

$$+ \frac{N^{N-k}}{(1-\alpha)^{N+k}} \frac{(N-1+\nu+k)!}{(N-1)!(N-1+\nu)!} \int_0^{(1-\alpha)^2/4\alpha} dt\, e^{-m^2Nt/(1-\alpha)}$$

$$\times \frac{t^{N-1}}{(t+1)^{N+k+\nu}} \left[4\alpha t - (1-\alpha)^2\right]^k$$

Microscopic limit revisited:

$$\mathcal{H}^{s}_{\nu,k}(\hat{\alpha},\hat{m}) = \frac{e^{-2\hat{\alpha}-\frac{\hat{m}^2}{8\hat{\alpha}}}}{4\hat{\alpha}} \int_0^{\hat{m}} du\, u^{k+1} K_{\nu}\left(\frac{\hat{m}u}{4\hat{\alpha}}\right) e^{-\frac{u^2}{8\hat{\alpha}}} I_{\nu+k}(u)$$

$$+ \frac{(4\hat{\alpha})^{\nu+k}}{2\hat{m}^{\nu}} \int_1^{\infty} ds\, e^{-\frac{\hat{m}^2}{8\hat{\alpha}s}-2\hat{\alpha}s} (1-s)^k s^{\nu-1}$$

Numerical results: random matrix simulations

Perform numerical simulations of random matrices:

$$D(\mu) = \begin{pmatrix} 0 & i\phi_1 + \mu\phi_2 \\ i\phi_1^\dagger + \mu\phi_2^\dagger & 0 \end{pmatrix},$$

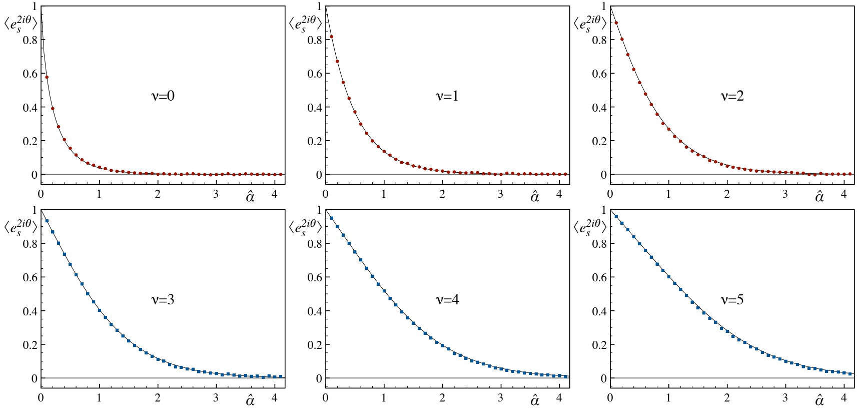
where ϕ_1, ϕ_2 are complex $(N+\nu) \times N$ matrices generated according to the Gaussian weight function:

$$w(X) \propto \exp(-N \operatorname{tr} X^\dagger X) = \prod_{k\ell} \exp\Big(-N(\operatorname{Re} X_{k\ell})^2\Big) \exp\Big(-N(\operatorname{Im} X_{k\ell})^2\Big).$$

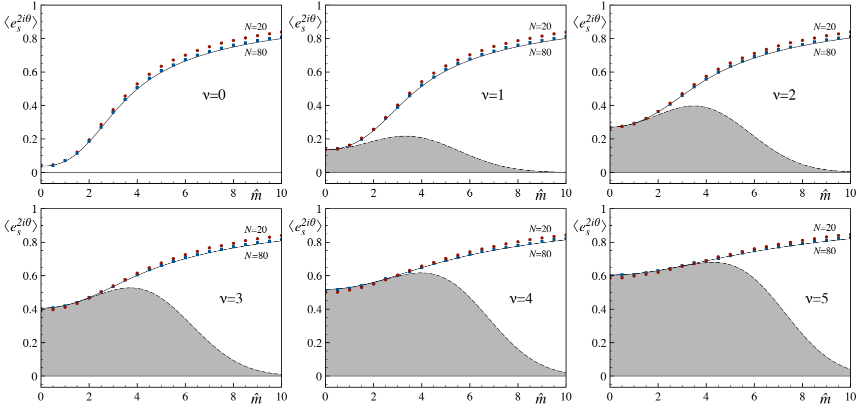
- real and imaginary parts of each matrix element are random numbers drawn from Gaussian distribution with standard deviation $1/\sqrt{2N}$.
- To investigate the microscopic limit ($N \to \infty$):
  keep $\hat{\alpha} = 2N\alpha$ and $\hat{m} = 2Nm$ constant, while varying N.

Results from analytic formula and from random matrix simulations

$\hat{\alpha}$-dependence in chiral limit ($\hat{m} = 0$)



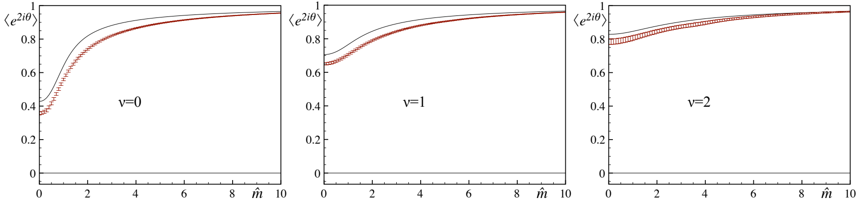
$\hat{m}$-dependence ($\hat{\alpha} = 1$)



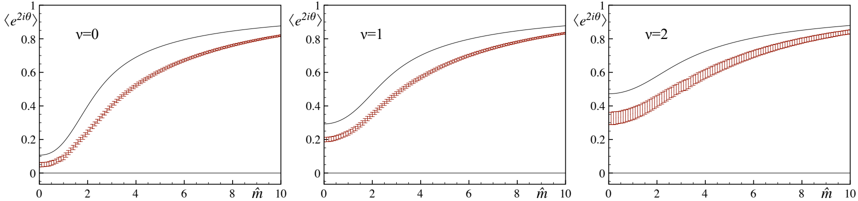
- shaded area: contribution of $u = 0$-singularity

Compare chRMT versus quenched LQCD

$\mu = 0.1\ (\hat{\alpha} = 0.175)$



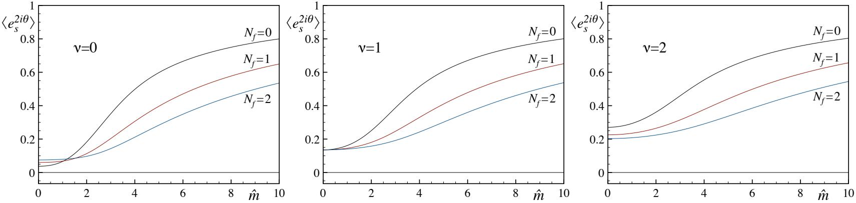
$\mu = 0.2\ (\hat{\alpha} = 0.615)$



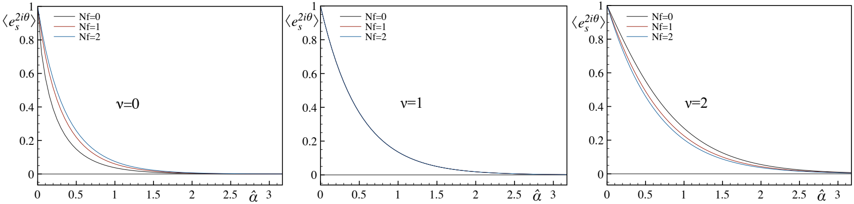
- No agreement $\longrightarrow$ not enough EV's in universality region

N_f-dependence of phase factor

$\hat{m}$-dependence ($\hat{\alpha} = 1$)



$\hat{\alpha}$-dependence in chiral limit ($\hat{m} = 0$)



Dynamical chRMT simulations

Introduce the weighted average of g wrt f:

$$\langle g \rangle_f = \frac{\int dx\, f(x) g(x)}{\int dx\, f(x)},$$

with $\langle 1 \rangle_f = 1$. Weighted averages are typical outcomes of importance samplings. From this we find relation between differently normalized integrals:

$$\langle f\, g \rangle_w = \langle g \rangle_{wf}\, \langle f \rangle_w\, .$$

So,

$$\begin{aligned} \langle \mathcal{O} \rangle_{N_f} &= \frac{\langle R e^{i\theta}\, \mathcal{O} \rangle_{N_f=0}}{\langle R e^{i\theta} \rangle_{N_f=0}} && \text{quenched with full reweighting} \\ &= \frac{\langle e^{i\theta}\, \mathcal{O} \rangle_R}{\langle e^{i\theta} \rangle_R} && \text{phase quenched with partial reweighting} \end{aligned}$$

Sign quenched with minimal reweighting

- Separate complex integral in real and imaginary part

$$\langle Re^{i\theta}\,\mathcal{O}\rangle_{N_f=0} = \langle R\cos\theta\;\mathcal{O}\rangle_{N_f=0} + i\langle R\sin\theta\;\mathcal{O}\rangle_{N_f=0}$$

- Sign quenched: Metropolis sampling of $R|\cos\theta|$ and $R|\sin\theta|$ to generate two independent Markov chains → maximize overlap with *significant* configurations
- Quenched and sign quenched averages are related by:

$$\langle R\cos\theta\;\mathcal{O}\rangle_{N_f=0} = \langle \mathrm{sgn}\cos\theta\;\mathcal{O}\rangle_{R|\cos\theta|}\langle R|\cos\theta|\rangle_{N_f=0}.$$

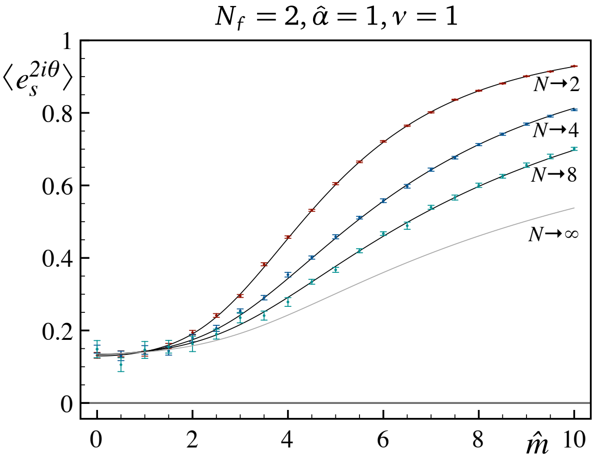
Hence,

$$\begin{aligned}\langle\mathcal{O}\rangle_{N_f} &= \frac{\langle Re^{i\theta}\;\mathcal{O}\rangle_{N_f=0}}{\langle Re^{i\theta}\rangle_{N_f=0}} \\ &= \frac{\langle \mathrm{sgn}\cos\theta\;\mathcal{O}\rangle_{R|\cos\theta|}\langle R|\cos\theta|\rangle_{N_f=0} + i\langle \mathrm{sgn}\sin\theta\;\mathcal{O}\rangle_{R|\sin\theta|}\langle R|\sin\theta|\rangle_{N_f=0}}{\langle \mathrm{sgn}\cos\theta\rangle_{R|\cos\theta|}\langle R|\cos\theta|\rangle_{N_f=0} + i\langle \mathrm{sgn}\sin\theta\rangle_{R|\sin\theta|}\langle R|\sin\theta|\rangle_{N_f=0}}\end{aligned}$$

As the partition function in the denominator has to be real this can be simplified to

$$\langle\mathcal{O}\rangle_{N_f} = \underbrace{\frac{1}{\langle \mathrm{sgn}\cos\theta\rangle_{R|\cos\theta|}}}_{\text{MC 1}}\left(\underbrace{\langle \mathrm{sgn}\cos\theta\;\mathcal{O}\rangle_{R|\cos\theta|}}_{\text{MC 1}} + i\underbrace{\langle \mathrm{sgn}\sin\theta\;\mathcal{O}\rangle_{R|\sin\theta|}}_{\text{MC 2}}\underbrace{\frac{\langle R|\sin\theta|\rangle_{N_f=0}}{\langle R|\cos\theta|\rangle_{N_f=0}}}_{\text{independent of }\mathcal{O}}\right)$$

Numerical results: unquenched case

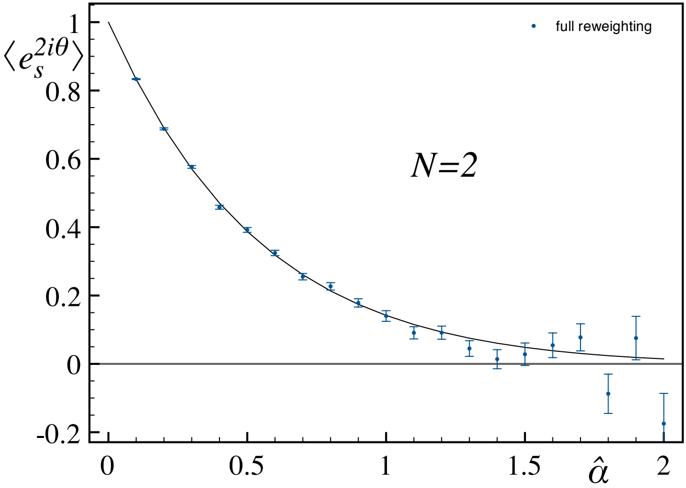


Finite-N predictions useful because:

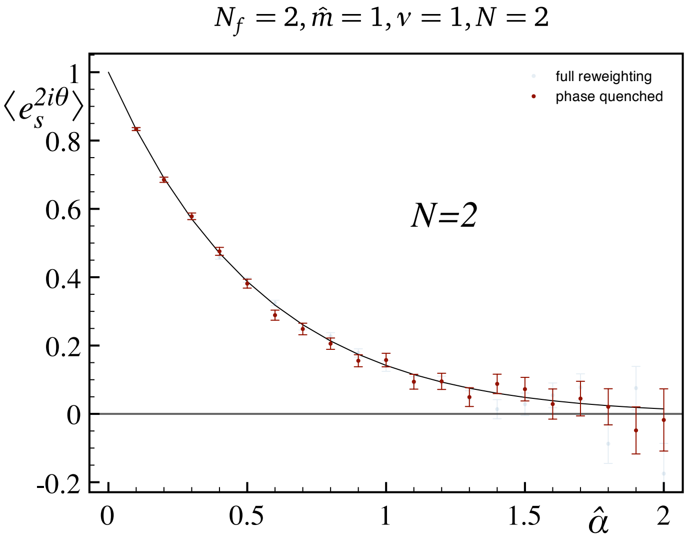
- convergence to microscopic limit is slower for $N_f \neq 0$
- dynamical simulations are expensive $\longrightarrow$ keep N small

Numerical results: unquenched case

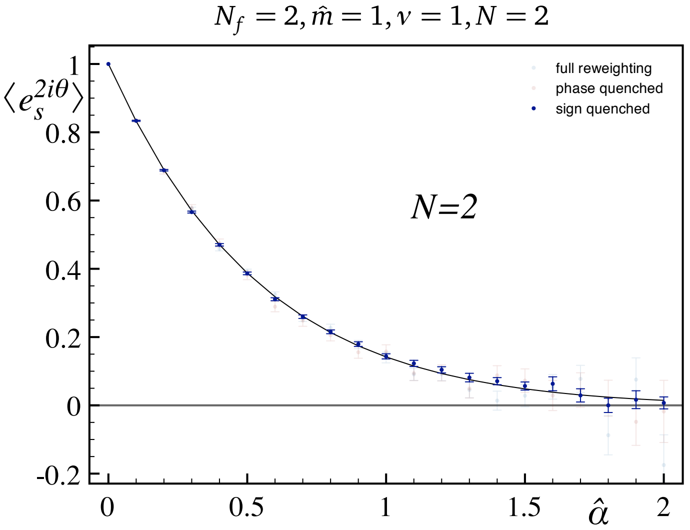
$N_f = 2, \hat{m} = 1, \nu = 1, N = 2$



Numerical results: unquenched case



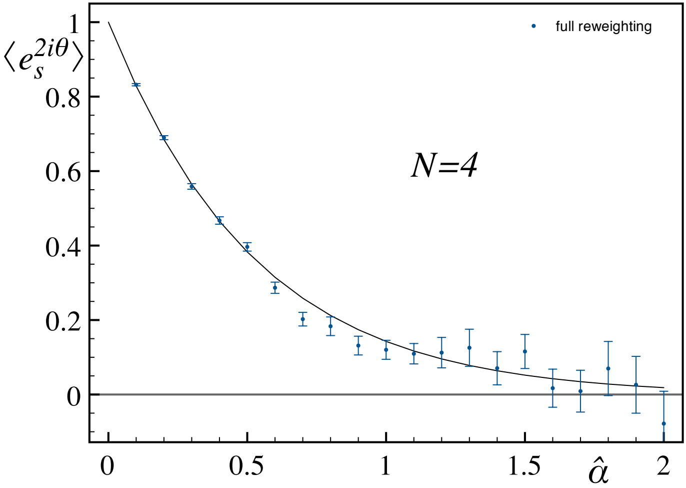
Numerical results: unquenched case



- 100,000 trial configs., 5,000-12,000 uncorrelated configs.

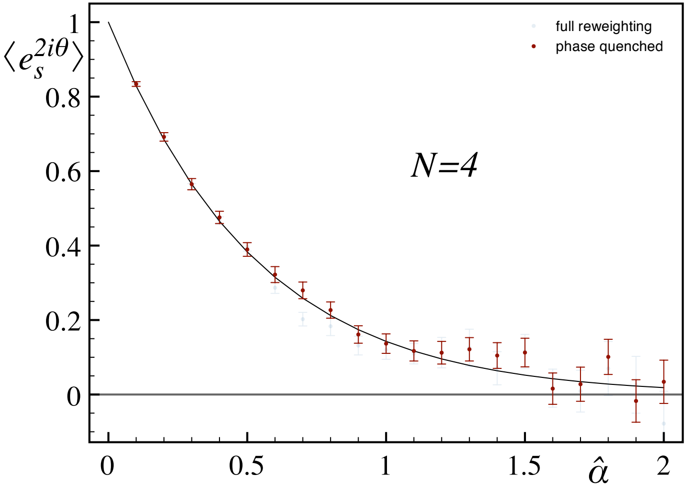
Numerical results: unquenched case

$$N_f = 2, \hat{m} = 1, \nu = 1, N = 4$$

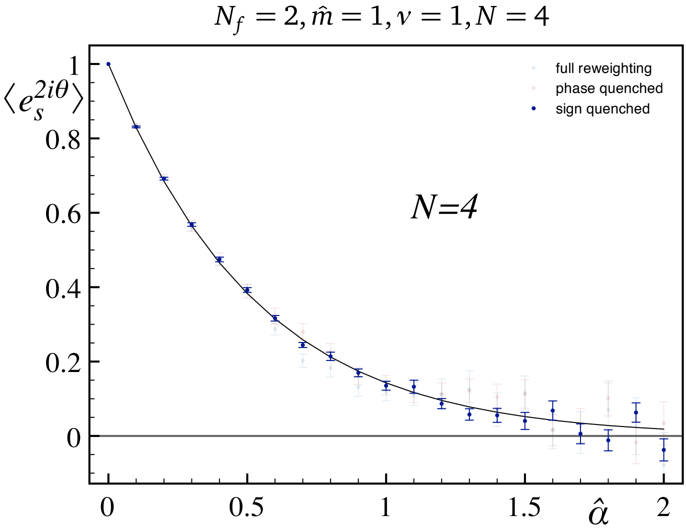


Numerical results: unquenched case

$$N_f = 2, \hat{m} = 1, \nu = 1, N = 4$$

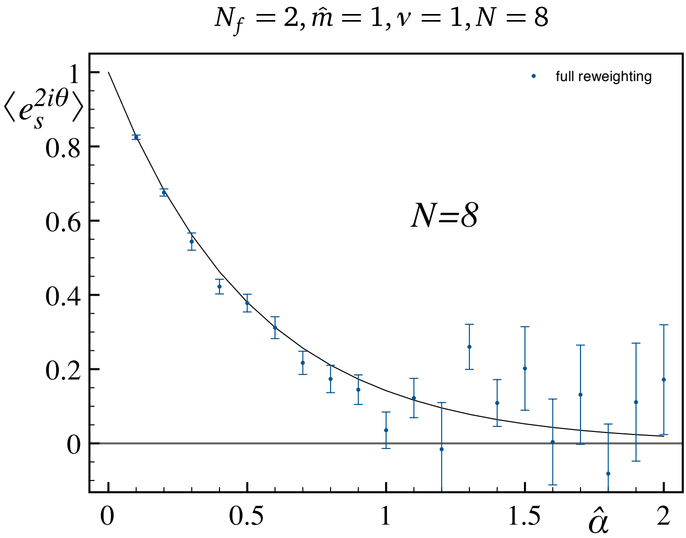


Numerical results: unquenched case

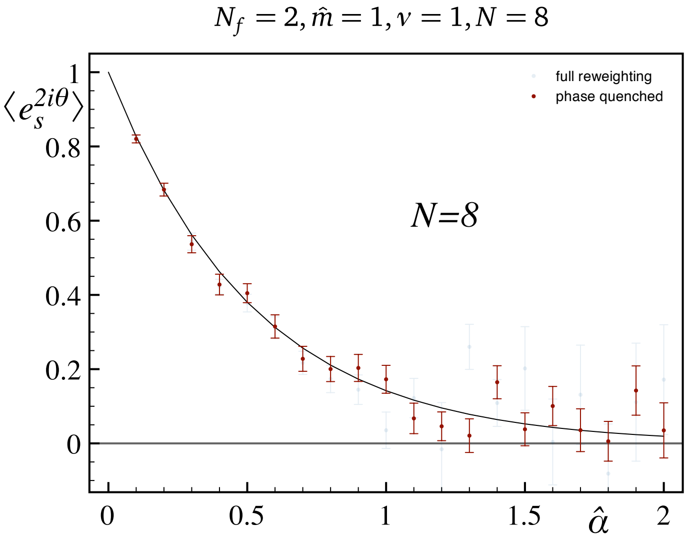


- 100,000 trial configs., 2,000-8,000 uncorrelated configs.

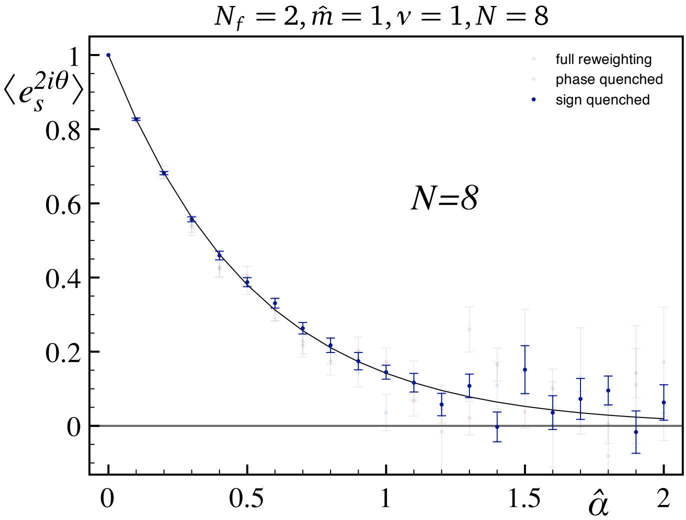
Numerical results: unquenched case



Numerical results: unquenched case



Numerical results: unquenched case



- Growing autocorrelations
- 100,000 trial configs., 1,000-5,000 uncorrelated configs.

Summary & Outlook

- Investigated equivalence between chRMT and LQCD at $\mu \neq 0$ for general topology
- Computed average phase factor using complex Cauchy transform of orthogonal polynomials for general topology
- Checked agreement with numerical random matrix simulations for:
  - quenched – microscopic limit
  - unquenched case – small N
- Used RMT simulations to test MCMC algorithms with complex action

- Apply overlap operator on larger lattices using our newly developed numerical methods
- Investigation of zero mode solutions at $\mu \neq 0$
- Improvement of sign quenched algorithm
- Implementation of sign quenched algorithm for other applications