STOCHASTIC QUANTIZATION AT FINITE CHEMICAL POTENTIAL: SIGN AND SILVER BLAZE PROBLEMS

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OUTLINE

- sign problem at finite chemical potential
- a revived approach: stochastic quantization
- three QCD inspired models
- relativistic Bose gas
- the Silver Blaze problem is not a problem

QCD AT FINITE μ

SIGN PROBLEM

- fermion determinant is complex

$$[\det M(\mu)]^* = \det M(-\mu)$$

- avoid fluctuating sign?

- simulations at $\mu = 0$ or with $|\det M(\mu)|$

but ...

QCD AT FINITE μ

SIGN PROBLEM

- important configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$
- cancelation between configurations with ‘positive’ and ‘negative’ weight
- how to pick the dominant configurations in the path integral?

QCD AT FINITE μ

SIGN PROBLEM

- important configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$
- cancelation between configurations with 'positive' and 'negative' weight
- how to pick the dominant configurations in the path integral?

radically different approach:

- complexifying all degrees of freedom: SU(3) $\rightarrow$ SL(3,$\mathbb{C}$)

stochastic quantization and complex Langevin dynamics based on

- with I.O.S.: stochastic quantization at finite chemical potential, 0807.1597 [hep-lat], JHEP
- can stochastic quantization evade the sign problem? – the relativistic Bose gas at finite chemical potential 0810.2089 [hep-lat], PRL
- complex Langevin dynamics at finite chemical potential: mean field analysis in the relativistic Bose gas, 0902.4686 [hep-lat]

more reading

- with I.O.S.: Lattice proceedings, 0809.5527 [hep-lat]
- SEWM proceedings: 0811.1850 [hep-ph]

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

field theory

Parisi & Wu '81

- path integral $Z = \int D\phi \, e^{-S}$
- Langevin dynamics in "fifth" time direction

$$\frac{\partial \phi(x,\theta)}{\partial \theta} = -\frac{\delta S[\phi]}{\delta \phi(x,\theta)} + \eta(x,\theta)$$

- Gaussian noise

$$\langle \eta(x,\theta) \rangle = 0 \qquad \langle \eta(x,\theta)\eta(x',\theta') \rangle = 2\delta(x-x')\delta(\theta-\theta')$$

- equilibrium distribution $P[\phi] \sim e^{-S}$

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

force $\partial S/\partial\phi$ complex: Parisi, Klauder '85

complexify Langevin dynamics

- example: real scalar field $\phi \to \phi^{\rm R} + i\phi^{\rm I}$
- Langevin eqs

$$\frac{\partial\phi^{\rm R}}{\partial\theta} = -{\rm Re}\,\frac{\delta S}{\delta\phi}\Big|_{\phi\to\phi^{\rm R}+i\phi^{\rm I}} + \eta$$
$$\frac{\partial\phi^{\rm I}}{\partial\theta} = -{\rm Im}\,\frac{\delta S}{\delta\phi}\Big|_{\phi\to\phi^{\rm R}+i\phi^{\rm I}}$$

- observables: analytic extension

$$\langle O(\phi)\rangle \to \langle O(\phi^{\rm R} + i\phi^{\rm I})\rangle$$

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

- associated Fokker-Planck equation

$$\frac{\partial P[\phi,\theta]}{\partial\theta} = \int d^dx \, \frac{\delta}{\delta\phi(x,\theta)} \left(\frac{\delta}{\delta\phi(x,\theta)} + \frac{\delta S[\phi]}{\delta\phi(x,\theta)} \right) P[\phi,\theta]$$

- stationary solution: $P[\phi] \sim e^{-S}$
- real action: formal proofs of convergence

$$P[\phi,\theta] = \frac{e^{-S[\phi]}}{Z} + \sum_{\lambda>0} e^{-\lambda\theta} P_\lambda[\phi]$$

- complex action: theoretical status less clear cut

but all other methods fail!

FINITE CHEMICAL POTENTIAL

TOWARDS QCD

consider three models with a partition function

$$Z = \int DU e^{-S_B} \det M \qquad \det M(\mu) = [\det M(-\mu)]^*$$

- QCD with static quarks
- SU(3) one link model
- U(1) one link model

observables:

- (conjugate) Polyakov loops
- density
- phase of determinant

THREE MODELS

I: QCD WITH STATIC QUARKS

$$Z = \int DU e^{-S_B} \det M$$

- bosonic action: standard SU(3) Wilson action

$$S_B = -\beta \sum_P \left(\frac{1}{6} \left[\mathrm{Tr}\, U_P + \mathrm{Tr}\, U_P^{-1} \right] - 1 \right)$$

- determinant $\det M$ for Wilson fermions

  fermion matrix:

$$M = 1 - \kappa \sum_{i=1}^{3} \text{space} - \kappa \left(e^{\mu} \Gamma_{+4} U_{x,4} T_4 + e^{-\mu} \Gamma_{-4} U_{x,4}^{-1} T_{-4} \right)$$

THREE MODELS

I: QCD WITH STATIC QUARKS

- hopping expansion:

$$\det M \approx \det\left[1-\kappa\left(e^{\mu}\Gamma_{+4}U_{x,4}T_4+e^{-\mu}\Gamma_{-4}U_{x,4}^{-1}T_{-4}\right)\right]$$
$$=\prod_{\mathbf{x}}\det\left(1+he^{\mu/T}\mathcal{P}_{\mathbf{x}}\right)^2\det\left(1+he^{-\mu/T}\mathcal{P}_{\mathbf{x}}^{-1}\right)^2$$

  with $h=(2\kappa)^{N_\tau}$ and (conjugate) Polyakov loops $\mathcal{P}_{\mathbf{x}}^{(-1)}$

- static quarks propagate in temporal direction only: Polyakov loops

- full gauge dynamics included

THREE MODELS

II: SU(3) ONE LINK MODEL

$$Z = \int dU\, e^{-S_B} \det M \qquad \text{link } U \in \text{SU(3)}$$

$$S_B = -\frac{\beta}{6}\left(\mathrm{Tr}\, U + \mathrm{Tr}\, U^{-1}\right)$$

determinant:

$$\begin{aligned}\det M &= \det\left[1 + \kappa\left(e^{\mu}\sigma_+ U + e^{-\mu}\sigma_- U^{-1}\right)\right] \\ &= \det\left(1 + \kappa e^{\mu} U\right) \det\left(1 + \kappa e^{-\mu} U^{-1}\right)\end{aligned}$$

with $\sigma_\pm = (\mathbb{1} \pm \sigma_3)/2$

- det in colour space remaining
- exact evaluation by integrating over the Haar measure

THREE MODELS

III: U(1) ONE LINK MODEL

U(1) model: link $U = e^{ix}$ with $-\pi < x \leq \pi$

$$S_B = -\frac{\beta}{2}\left(U + U^{-1}\right) = -\beta \cos x$$

determinant:

$$\det M = 1 + \frac{1}{2}\kappa\left[e^{\mu}U + e^{-\mu}U^{-1}\right] = 1 + \kappa \cos(x - i\mu)$$

partition function:

$$Z = \int_{-\pi}^{\pi} \frac{dx}{2\pi}\, e^{\beta \cos x}\left[1 + \kappa \cos(x - i\mu)\right]$$

- all observables can be computed analytically

COMPLEX LANGEVIN DYNAMICS

Langevin update:

$$U(\theta+\epsilon) = R(\theta)\,U(\theta) \qquad\qquad R = \exp\left[i\lambda_a\left(\epsilon K_a + \sqrt{\epsilon}\eta_a\right)\right]$$

- drift term

$$K_a = -D_a S_{\rm eff} \qquad\qquad S_{\rm eff} = S_B + S_F \qquad\qquad S_F = -\ln\det M$$

- noise

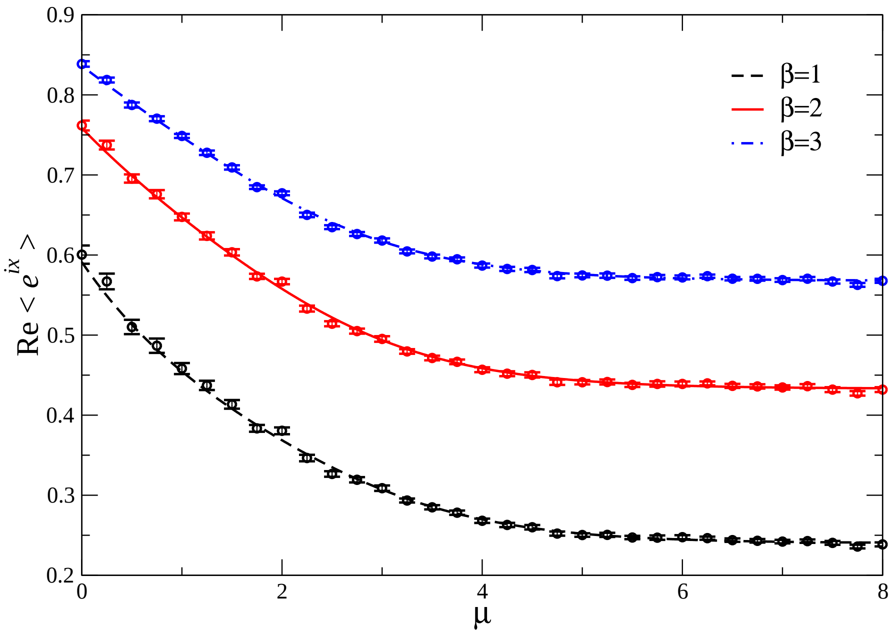
$$\langle\eta_a\rangle = 0 \qquad\qquad \langle\eta_a\eta_b\rangle = 2\delta_{ab}$$

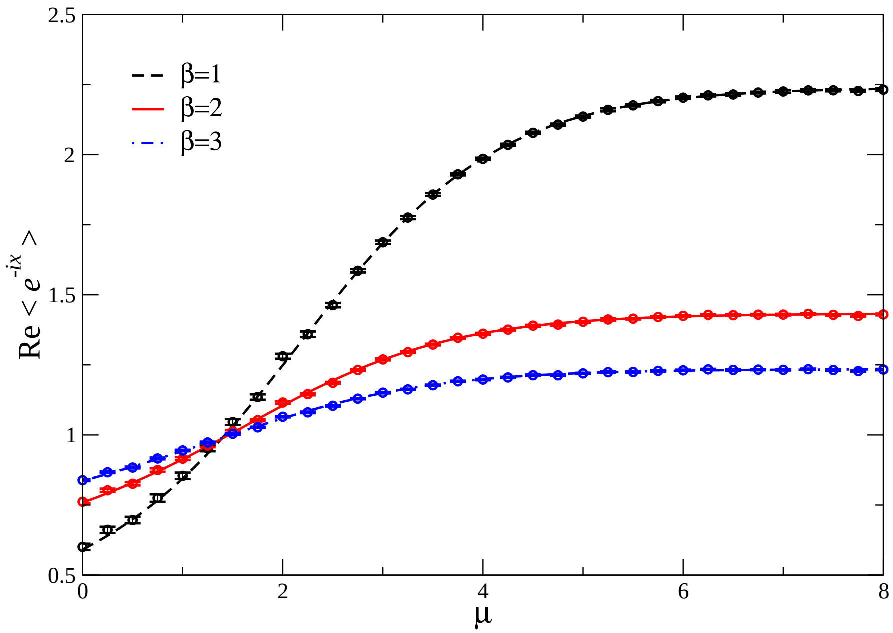
real action: $\Rightarrow K^\dagger = K \Leftrightarrow U \in$ SU(3)

complex action: $\Rightarrow K^\dagger \neq K \Leftrightarrow U \in$ SL($3,\mathbb{C}$)

(CONJUGATE) POLYAKOV LOOPS

U(1) ONE LINK MODEL



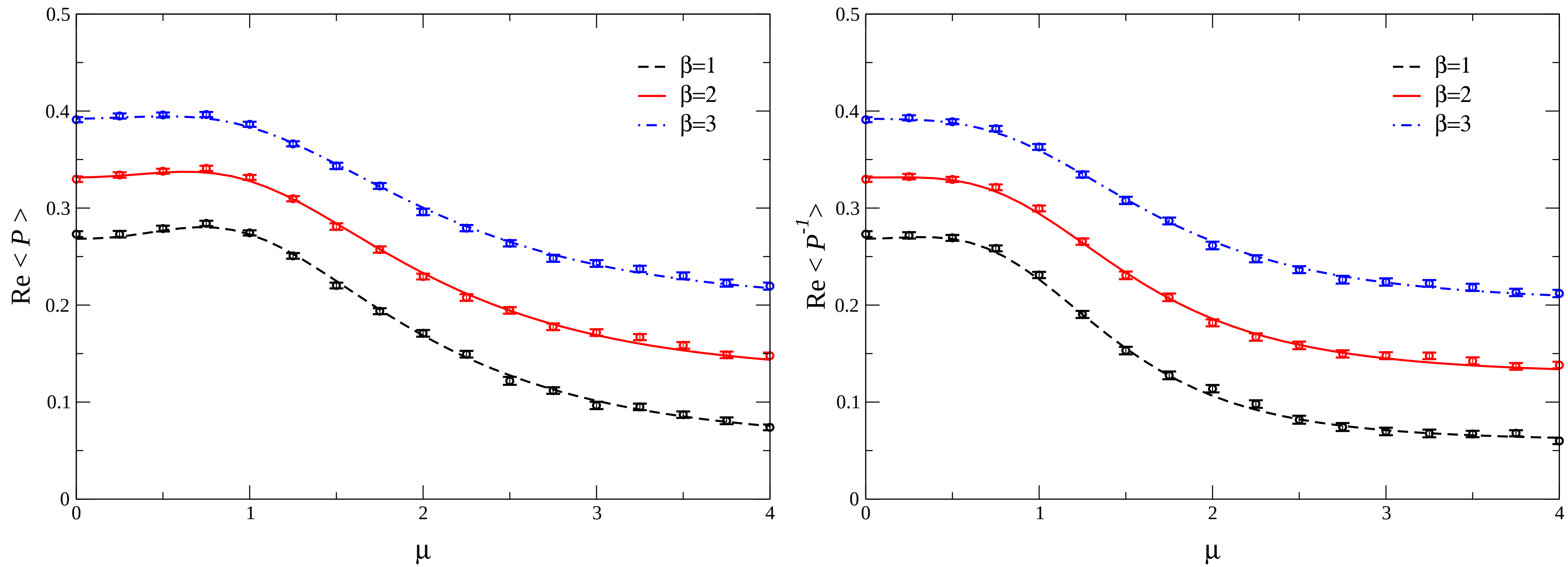


- data points: complex Langevin
  stepsize $\epsilon = 5 \times 10^{-5}$, 5×10^7 time steps
- lines: exact results

excellent agreement for all μ

(CONJUGATE) POLYAKOV LOOPS

SU(3) ONE LINK MODEL



- data points: complex Langevin
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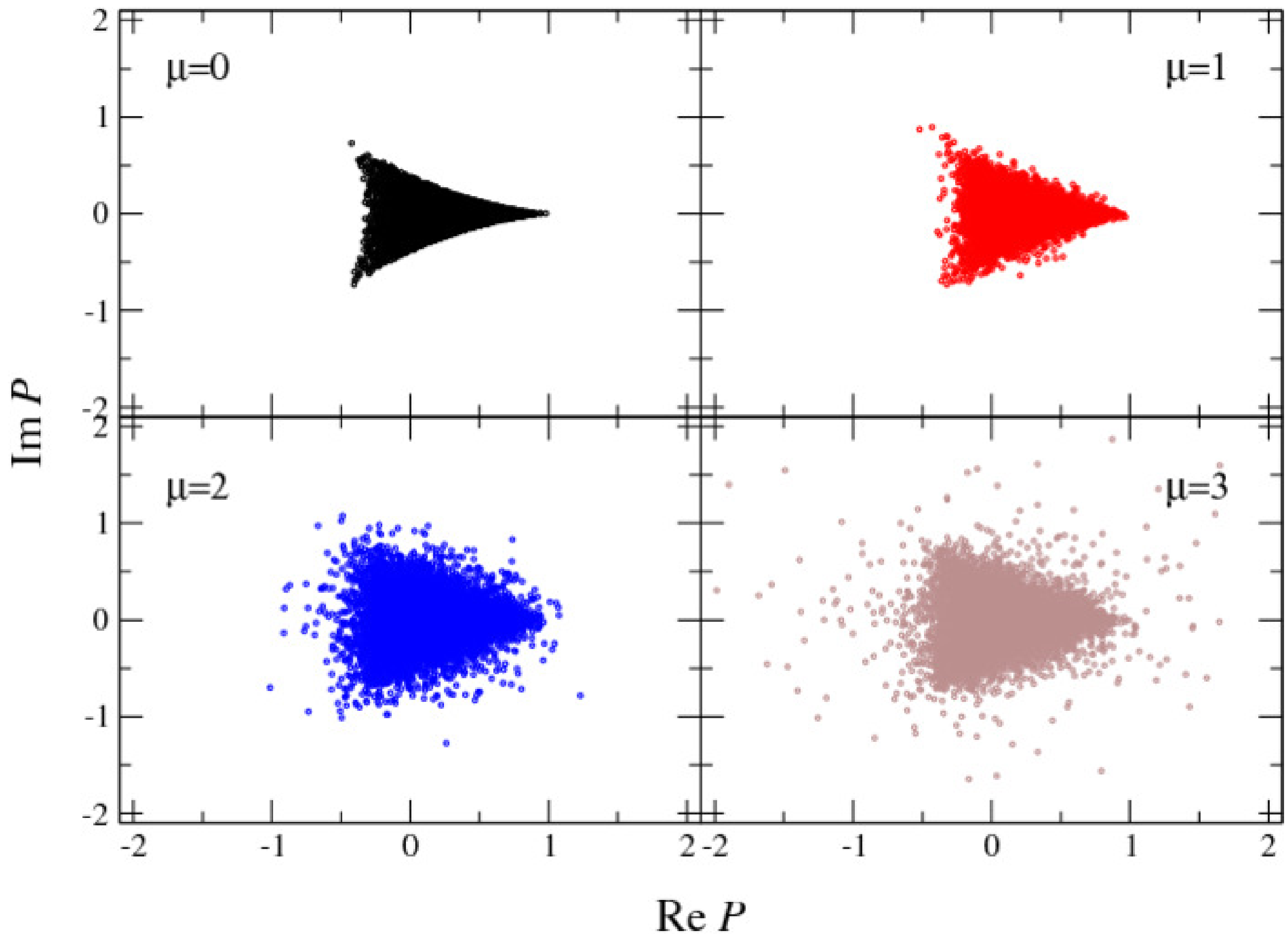
SU(3) ONE LINK MODEL

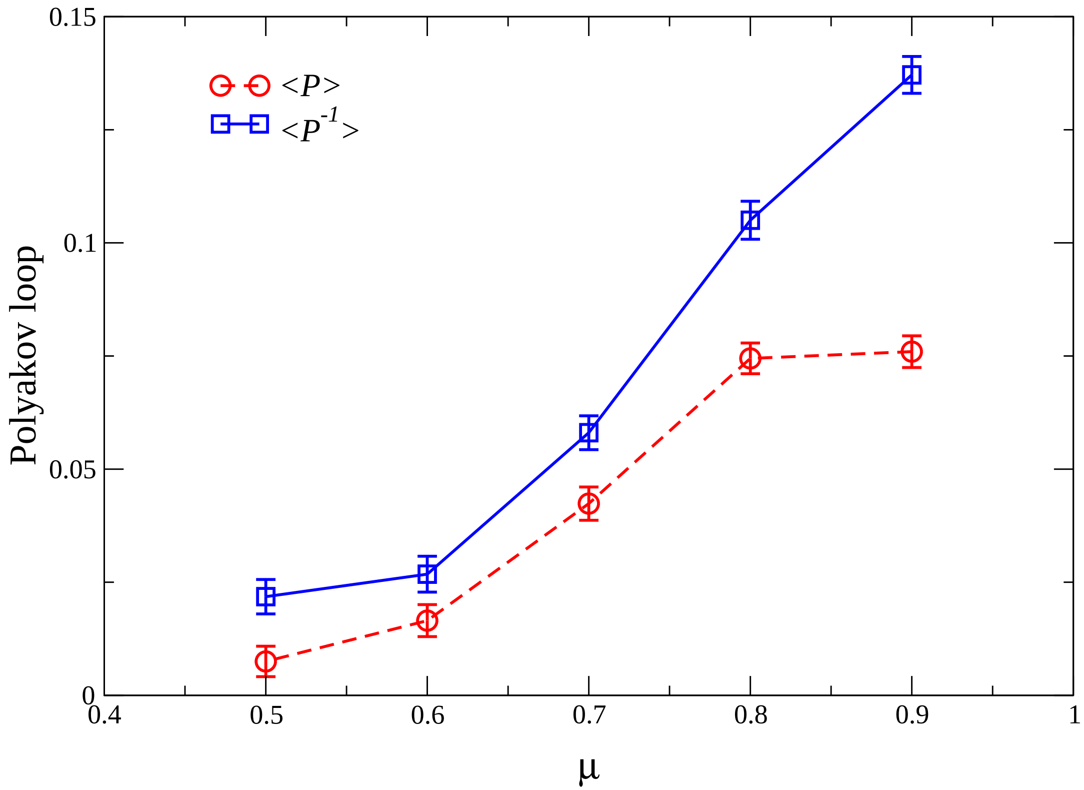


scatter plot of P during Langevin evolution

(CONJUGATE) POLYAKOV LOOPS

QCD WITH STATIC QUARKS

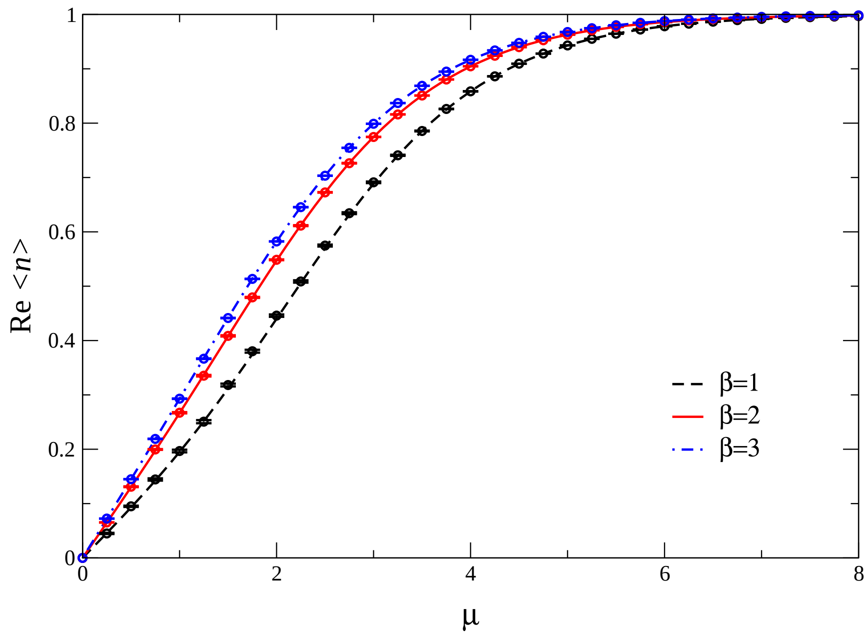
first results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$



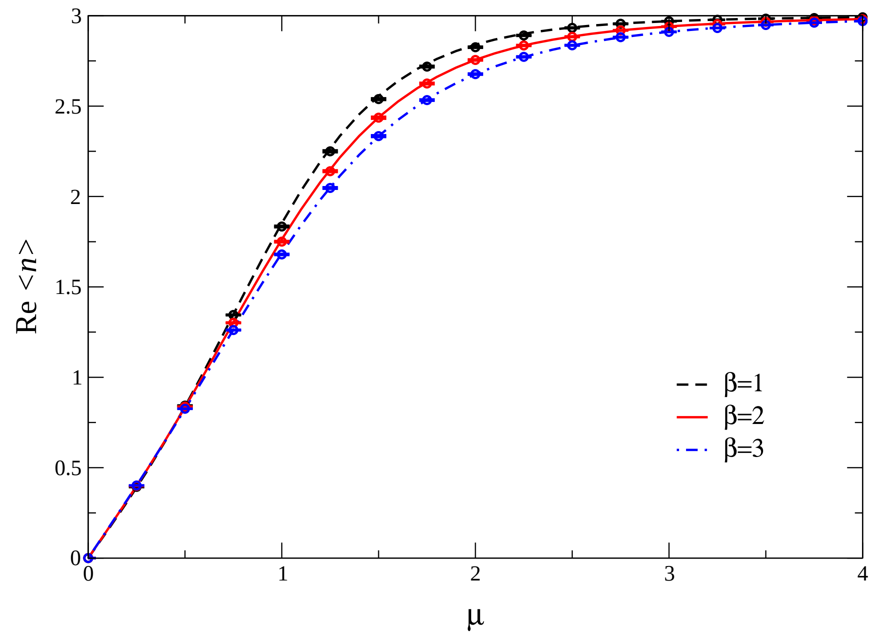
low-density “confining” phase $\Rightarrow$ high-density “deconfining” phase

DENSITY

U(1) ONE LINK MODEL



SU(3) ONE LINK MODEL

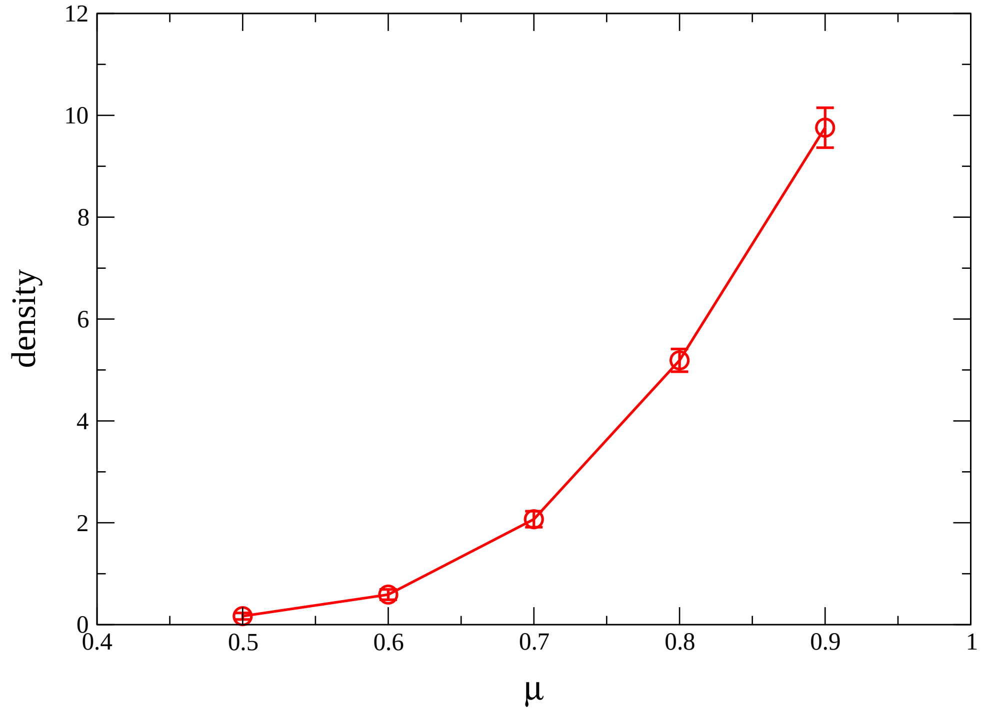


- linear increase at small μ
- saturation at large μ

excellent agreement for all μ

DENSITY

QCD WITH STATIC QUARKS



first results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density phase $\Rightarrow$ high-density phase

$\mathbf{SU(3) \to SL(3,\mathbb{C})}$

QCD WITH STATIC QUARKS

- complex Langevin dynamics: no longer in SU(3)
- instead $U \in$ SL(3,$\mathbb{C}$)
- in terms of gauge potentials $U = e^{i\lambda_a A_a/2}$
  A_a is now complex
- how far from SU(3)?

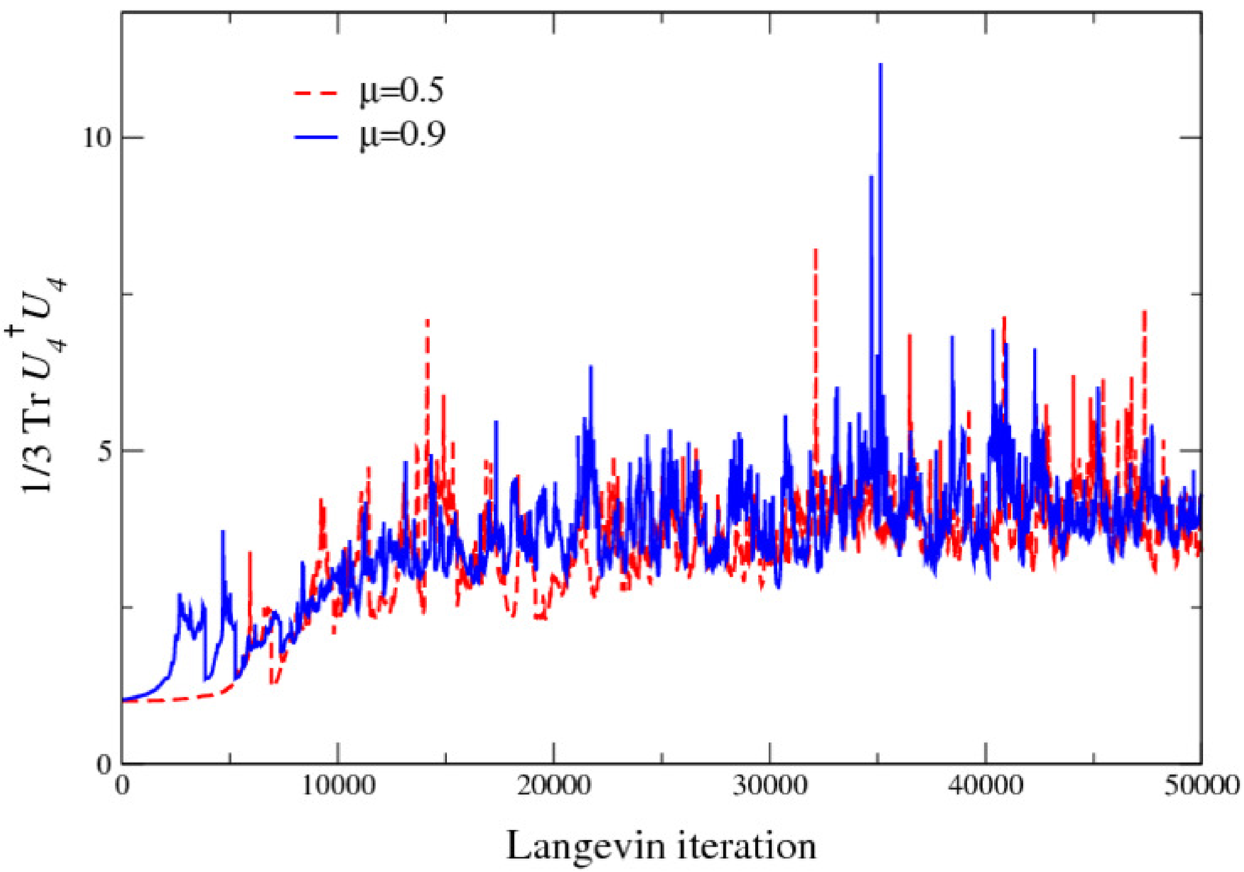
consider

$$\frac{1}{N}\mathrm{Tr}\, U^\dagger U \begin{cases} = 1 \;\; \text{if} \;\; U \in \text{SU}(N) \\ \\ \geq 1 \;\; \text{if} \;\; U \in \text{SL}(N,\mathbb{C}) \end{cases}$$

$\mathbf{SU(3)} \rightarrow \mathbf{SL(3,\mathbb{C})}$

QCD WITH STATIC QUARKS

$$\frac{1}{3}\mathrm{Tr}\, U^\dagger U \geq 1 \qquad\qquad = 1 \text{ if } U \in \mathrm{SU(3)}$$



COMPLEXIFICATION OF PHASE SPACE

WHY DOES IT WORK?

- most approaches start from $\mu = 0$ or $|\det M(\mu)|$
- complex Langevin dynamics radically different

$\Rightarrow$ complexification of degrees of freedom

visualization in U(1) model

- understanding in terms of classical fixed points

CLASSICAL FLOW

U(1) ONE LINK MODEL

- link $U = e^{ix}$ complexification $x \to z = x + iy$
- Langevin dynamics:

$$\dot{x} = K_x + \eta \qquad \dot{y} = K_y$$

- classical forces:

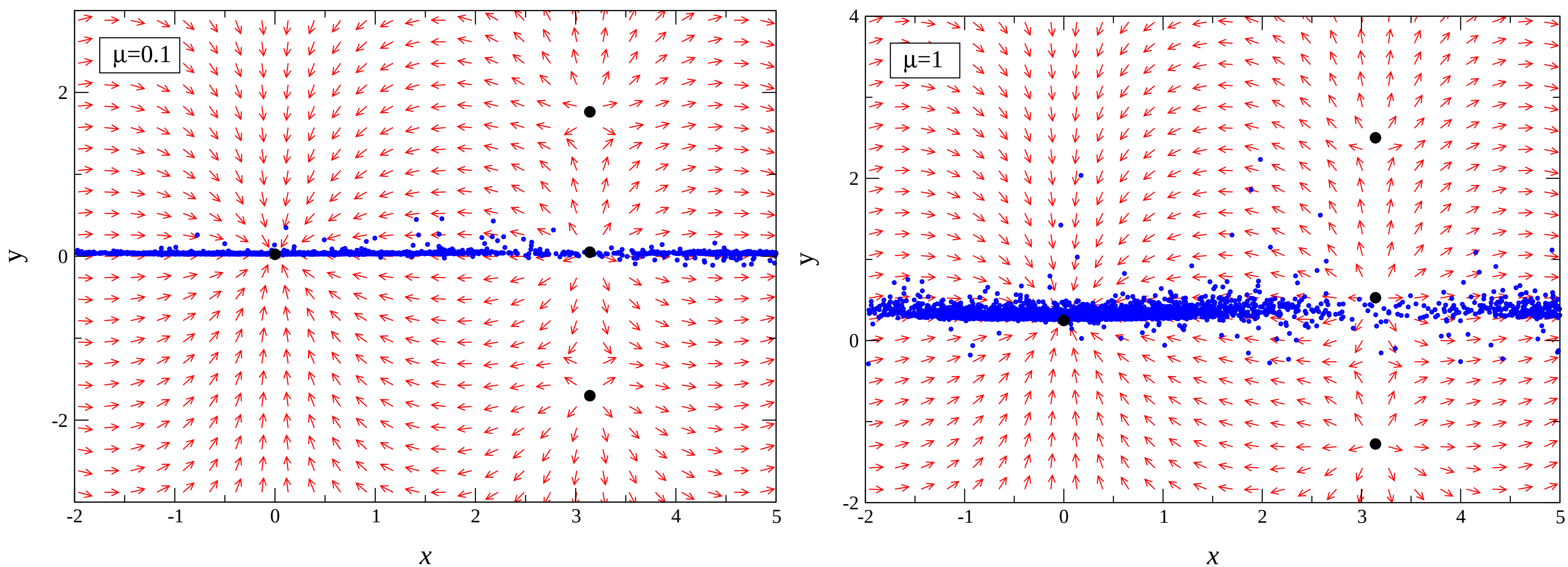
$$K_x = -\mathrm{Re}\frac{\partial S}{\partial x}\Big|_{x\to z} \qquad K_y = -\mathrm{Im}\frac{\partial S}{\partial x}\Big|_{x\to z}$$

- classical fixed points: $K_x = K_y = 0$

CLASSICAL FLOW

U(1) ONE LINK MODEL

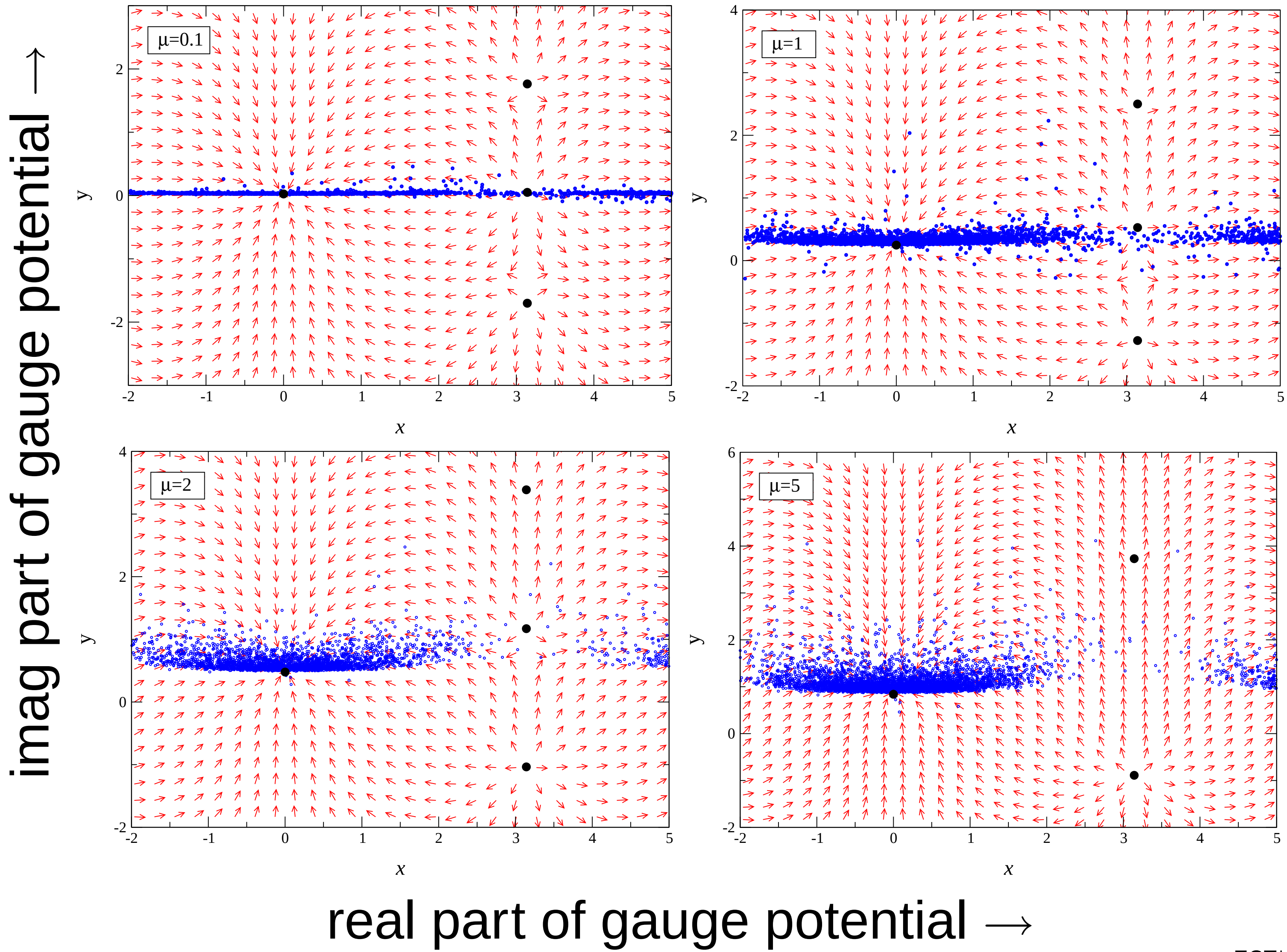
flow diagrams and Langevin evolution



- black dots: classical fixed points
- $\mu = 0$: dynamics only in x direction
- $\mu > 0$: spread in y direction

CLASSICAL FLOW

U(1) ONE LINK MODEL



PHASE TRANSITIONS AND THE SILVER BLAZE

intruiging questions:

- how severe is the sign problem?
- thermodynamic limit?
- phase transitions?
- Silver Blaze problem? Cohen '03
- ...

study in a model with a phase diagram with similar features as QCD at low temperature

$\Rightarrow$ relativistic Bose gas at nonzero μ or scalar O(2) model

RELATIVISTIC BOSE GAS AT NONZERO μ

PHASE TRANSITIONS AND THE SILVER BLAZE

- continuum action

$$S = \int d^4x \Big[\, |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu \left(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi\right) + \lambda |\phi|^4 \Big]$$

- complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu)$ as in QCD

RELATIVISTIC BOSE GAS AT NONZERO μ

PHASE TRANSITIONS AND THE SILVER BLAZE

- lattice action

$$S = \sum_x \Big[\left(2d + m^2\right) \phi_x^* \phi_x + \lambda \left(\phi_x^* \phi_x\right)^2 - \sum_{\nu=1}^{4} \left(\phi_x^* e^{-\mu\delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}}^* e^{\mu\delta_{\nu,4}} \phi_x \right) \Big]$$

- complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu)$ as in QCD

RELATIVISTIC BOSE GAS AT NONZERO μ

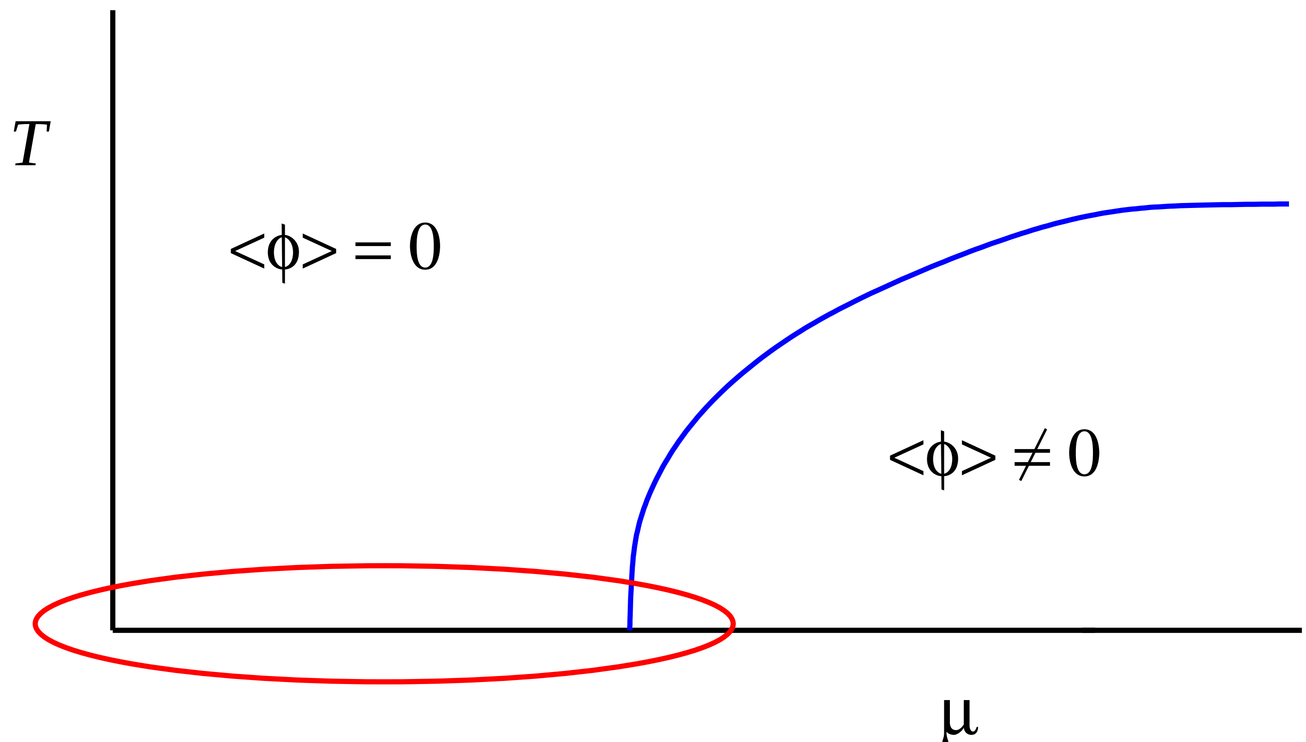
PHASE TRANSITIONS AND THE SILVER BLAZE

tree level potential in the continuum

$$V(\phi) = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

condensation when $\mu^2 > m^2$, SSB

Silver
Blaze
problem



RELATIVISTIC BOSE GAS AT NONZERO μ

COMPLEX LANGEVIN

- write $\phi = (\phi_1 + i\phi_2)/\sqrt{2} \;\Rightarrow\; \phi_a \;\; (a = 1, 2)$
- complexification $\phi_a \;\rightarrow\; \phi_a^{\rm R} + i\phi_a^{\rm I}$
- complex Langevin equations

$$\frac{\partial \phi_a^{\rm R}}{\partial \theta} = -{\rm Re}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a \to \phi_a^{\rm R} + i\phi_a^{\rm I}} + \eta_a$$

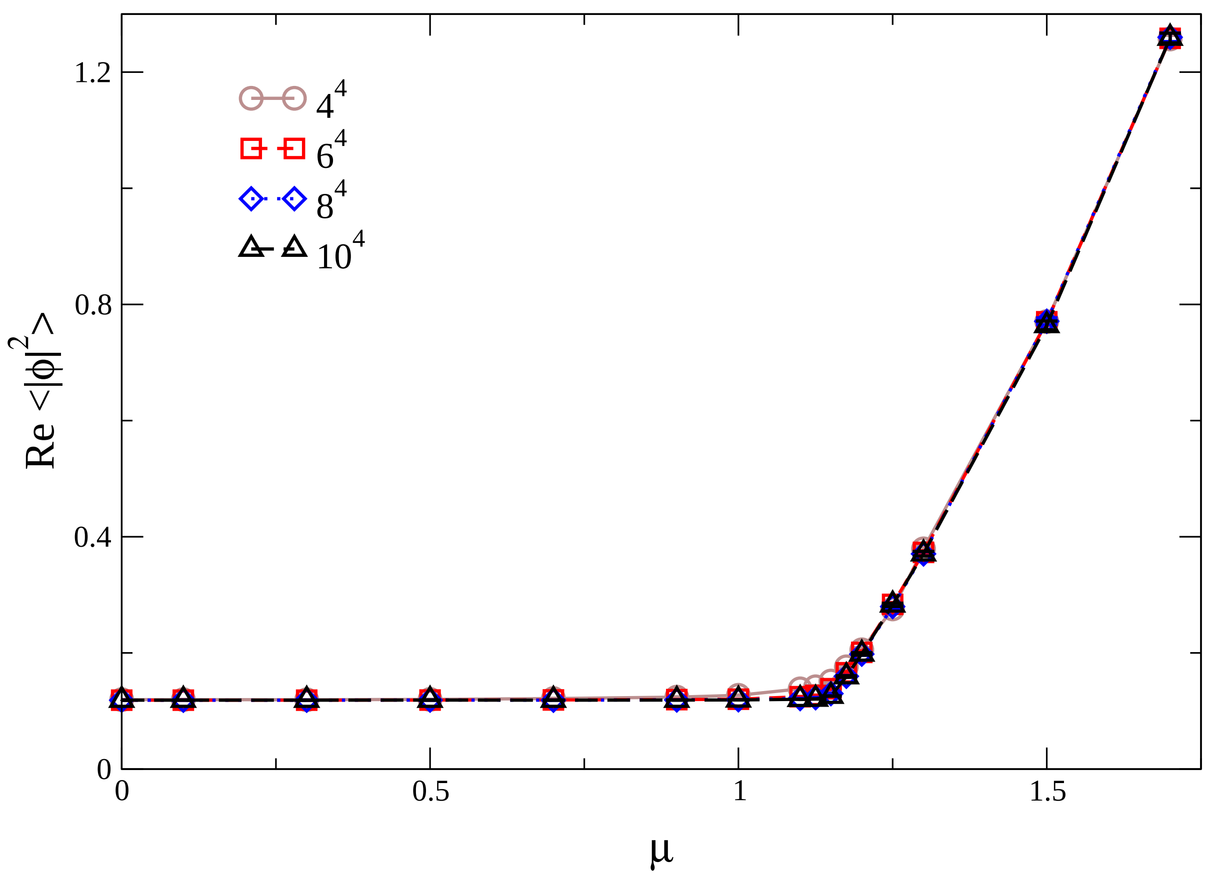
$$\frac{\partial \phi_a^{\rm I}}{\partial \theta} = -{\rm Im}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a \to \phi_a^{\rm R} + i\phi^{\rm I}}$$

- straightforward to solve numerically, $m = \lambda = 1$
- lattices of size N^4, with $N = 4, 6, 8, 10$
- no instabilities etc

RELATIVISTIC BOSE GAS

COMPLEX LANGEVIN

field modulus squared $$|\phi|^2 \rightarrow \frac{1}{2}\left({\phi_a^{\rm R}}^2 - {\phi_a^{\rm I}}^2\right) + i\phi_a^{\rm R}\phi_a^{\rm I}$$

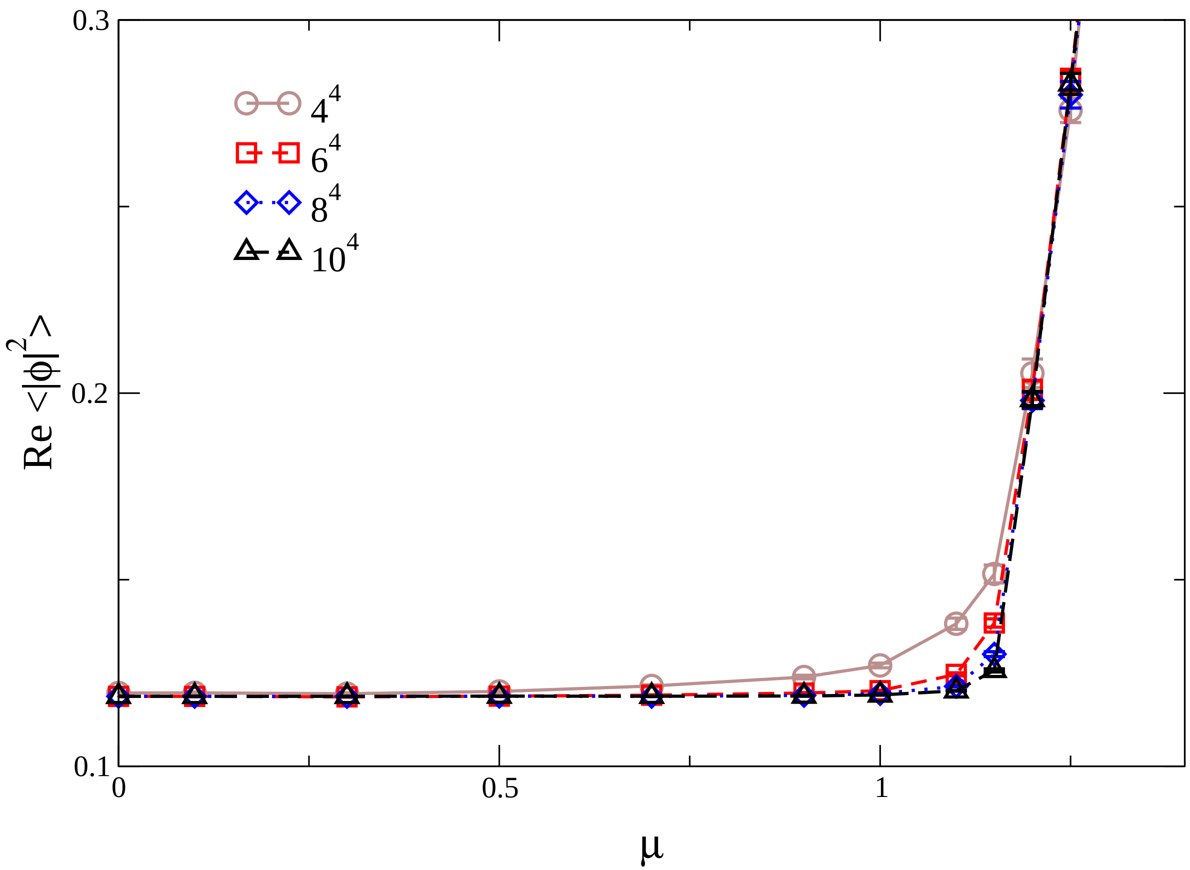


Silver Blaze!

RELATIVISTIC BOSE GAS

COMPLEX LANGEVIN

field modulus squared $$|\phi|^2 \to \frac{1}{2}\left({\phi_a^{\rm R}}^2 - {\phi_a^{\rm I}}^2\right) + i\phi_a^{\rm R}\phi_a^{\rm I}$$

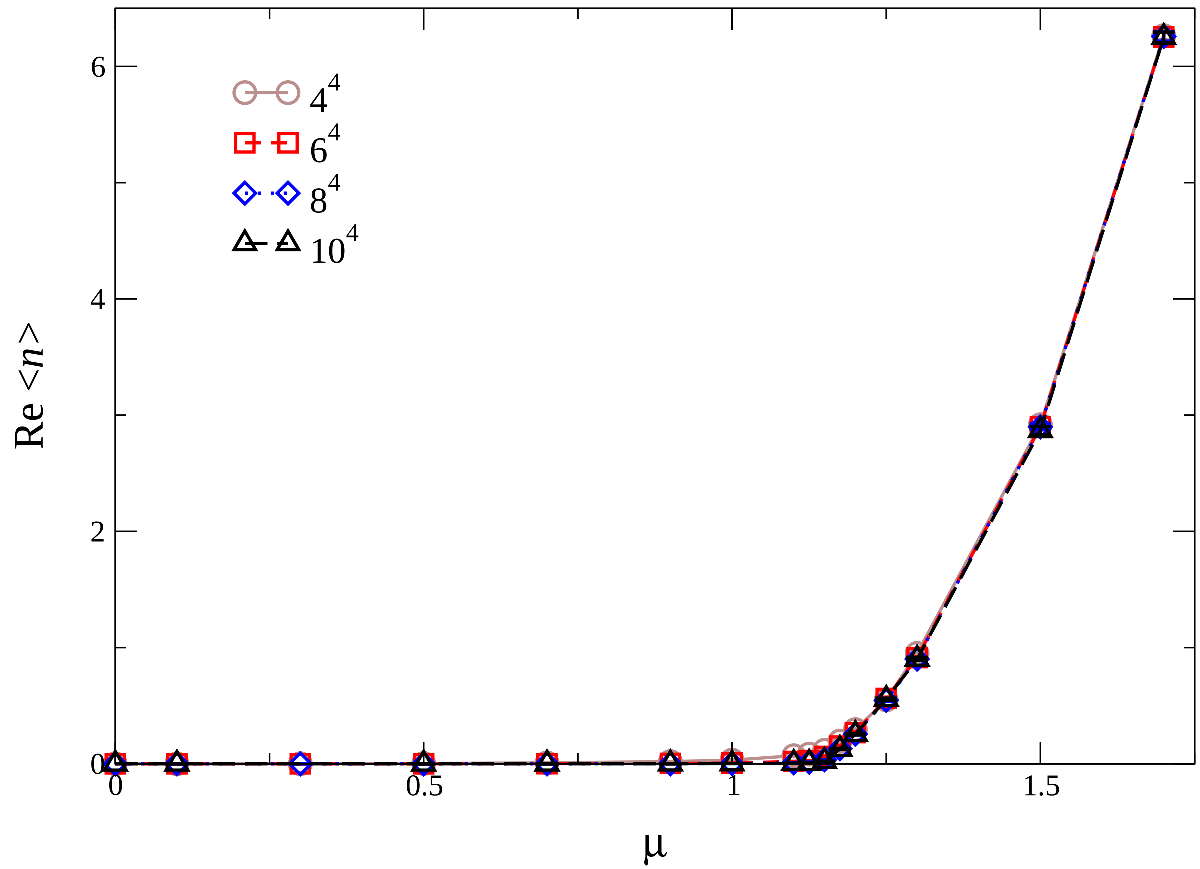


second order phase transition in thermodynamic limit

RELATIVISTIC BOSE GAS

COMPLEX LANGEVIN

density $\langle n \rangle = (1/\Omega)\partial \ln Z/\partial\mu$

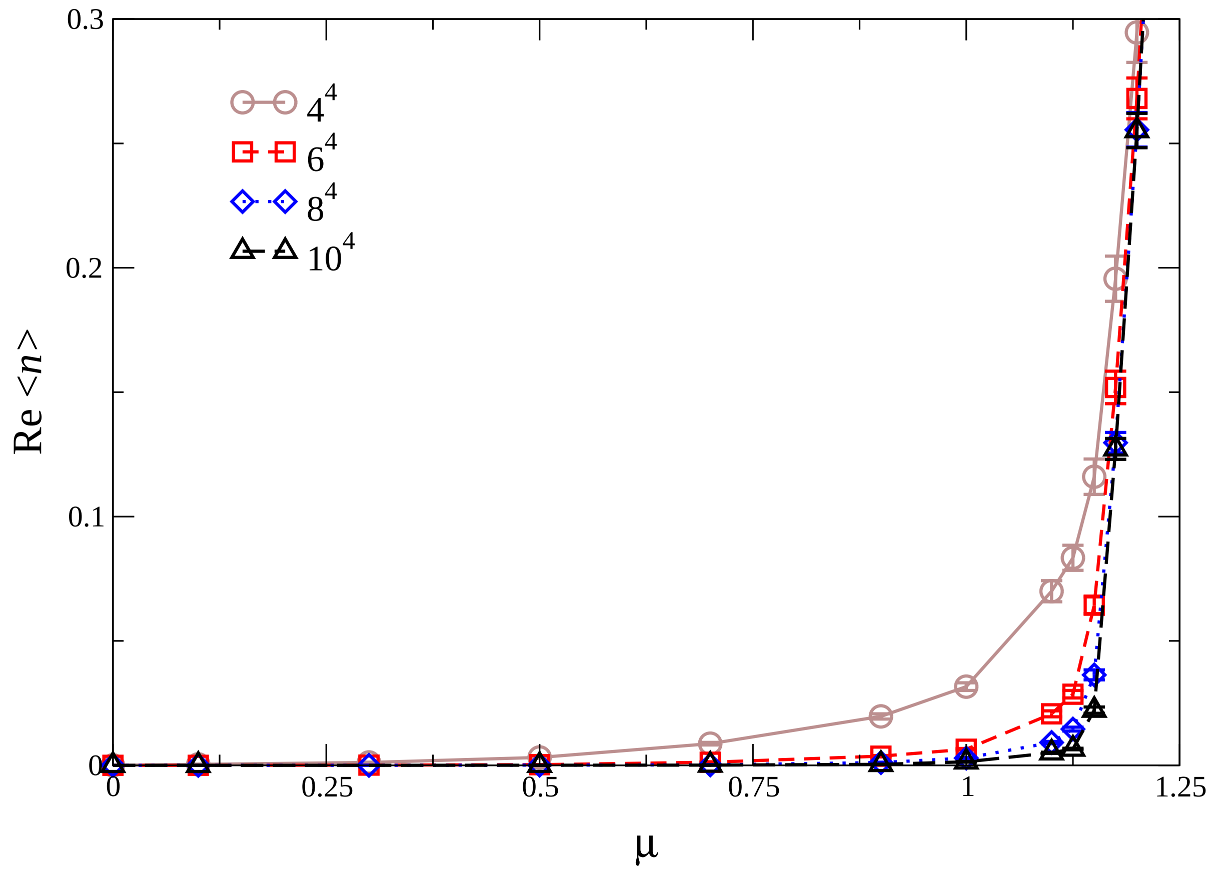


Silver Blaze

RELATIVISTIC BOSE GAS

COMPLEX LANGEVIN

density $\langle n \rangle = (1/\Omega)\partial \ln Z/\partial\mu$



second order phase transition in thermodynamic limit

Silver Blaze and the sign problem

relativistic Bose gas

Silver Blaze and sign problems are intimately related

- complex action

$$e^{-S} = |e^{-S}|e^{i\varphi}$$

- phase quenched theory

$$Z_{\text{pq}} = \int D\phi |e^{-S}|$$

different physics

QCD: phase quenched = finite isospin chemical potential

different onset: $m_N/3$ versus $m_\pi/2$

SILVER BLAZE AND THE SIGN PROBLEM

PHASE QUENCHED

phase quenched theory in this case:

- real action
- chemical potential appears only in the mass parameter (in continuum notation)

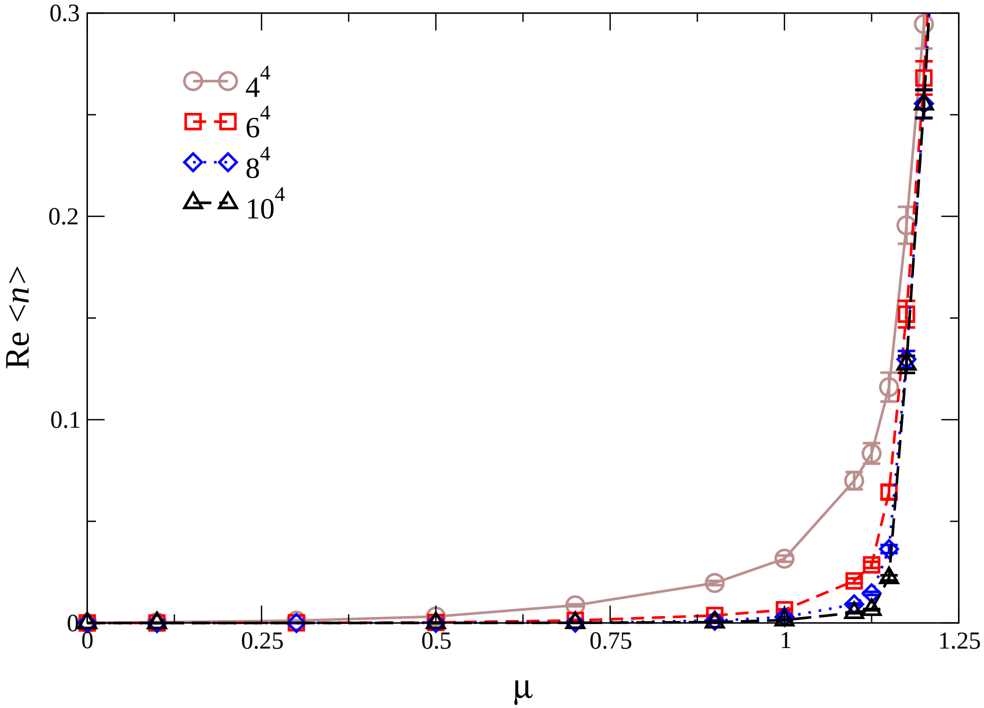
$$V = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

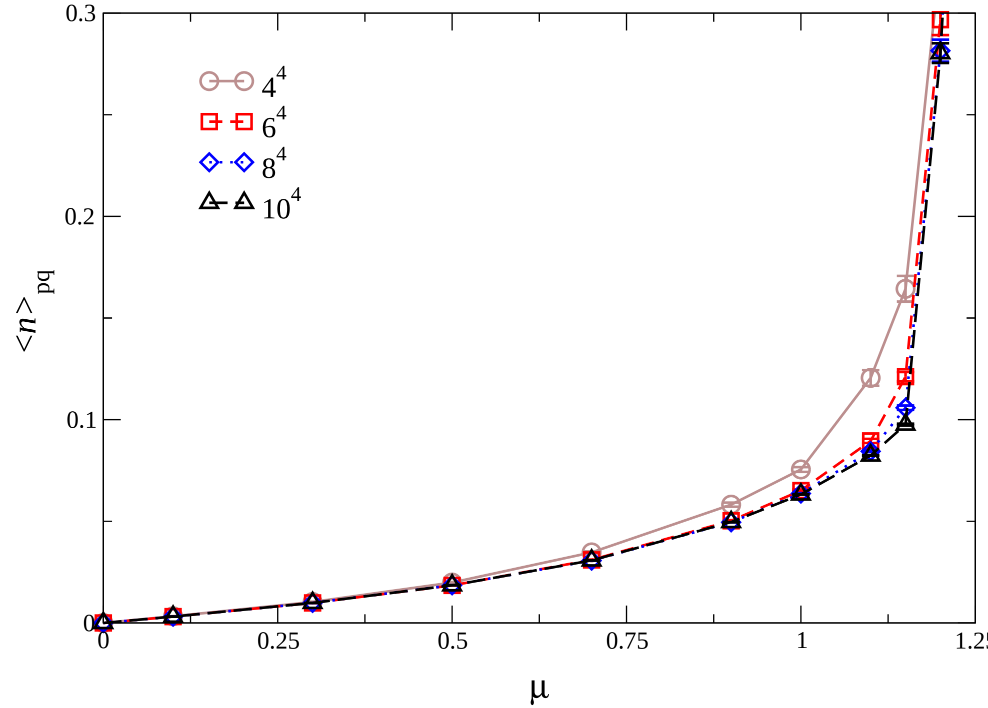
- dynamics of symmetry breaking, no Silver Blaze

SILVER BLAZE AND THE SIGN PROBLEM

COMPLEX VS PHASE QUENCHED

density





complex phase quenched

phase $e^{i\varphi} = e^{-S}/|e^{-S}|$ does precisely what is expected

HOW SEVERE IS THE SIGN PROBLEM?

AVERAGE PHASE FACTOR

- complex action $e^{-S} = |e^{-S}|e^{i\varphi}$
- full and phase quenched partition functions

$$Z_{\rm full} = \int D\phi \, e^{-S} \qquad\qquad Z_{\rm pq} = \int D\phi |e^{-S}|$$

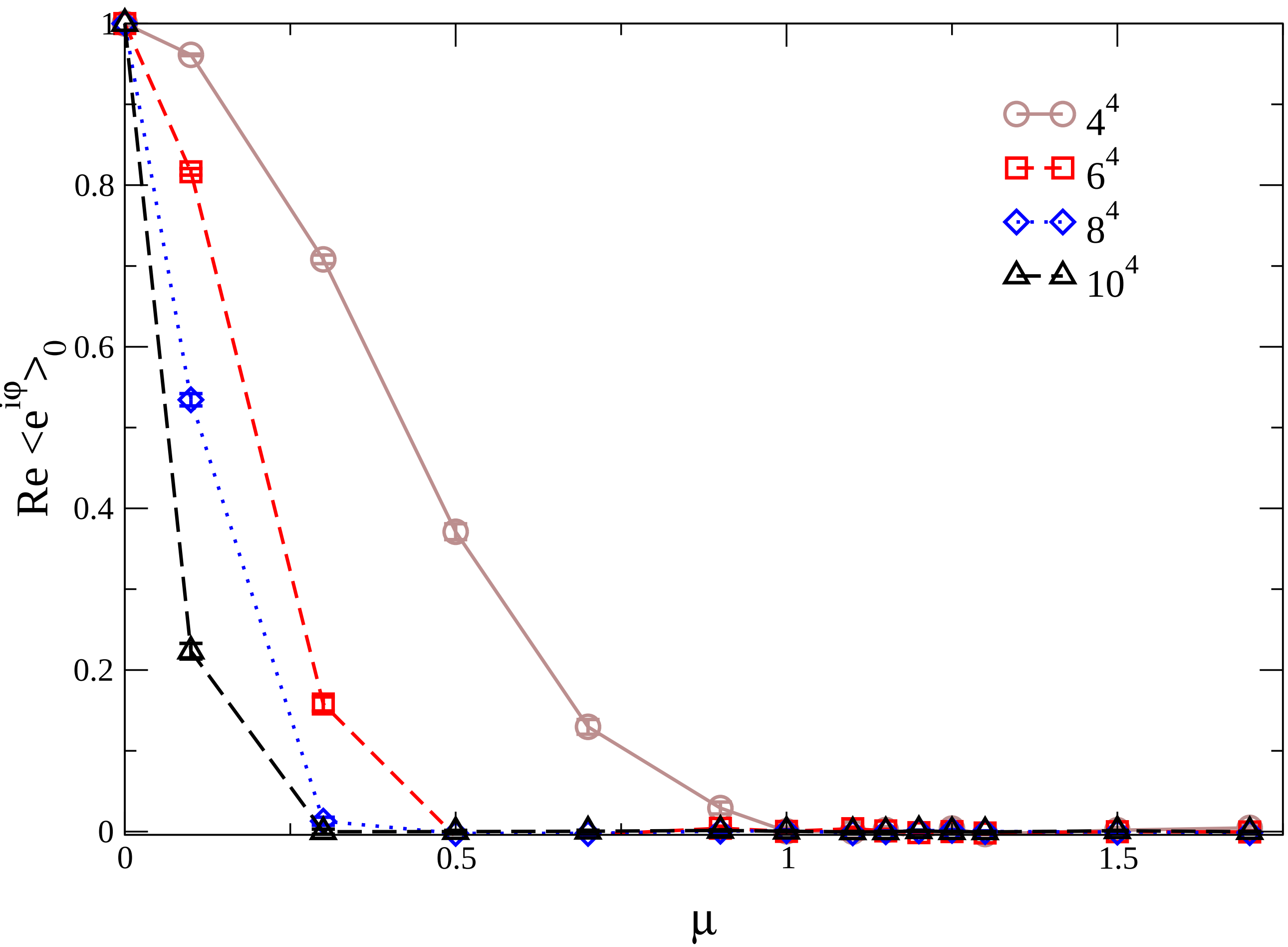
- average phase factor in phase quenched theory

$$\langle e^{i\varphi}\rangle_{\rm pq} = \frac{Z_{\rm full}}{Z_{\rm pq}} = e^{-\Omega\Delta f} \to 0 \quad \text{as} \quad \Omega \to \infty$$

- exponentially hard in thermodynamic limit

How severe is the sign problem?

AVERAGE PHASE FACTOR



average phase factor $\langle e^{i\varphi} \rangle_{\mathrm{pq}}$

HOW SEVERE IS THE SIGN PROBLEM?

AVERAGE PHASE FACTOR

- phase factor behaves exactly as expected
- for larger μ: phase factor $\to 0$ on all volumes
  - in the condensed phase: phase factor $= 0$
- at small μ, sign problem gets exponentially worse with increasing volume

yet, no problem in practice

RELATIVISTIC BOSE GAS

ANALYTICAL INSIGHT

- free Langevin dynamics
- real Fokker-Planck distribution
- include interactions with mean field approximation

see 0902.4686 [hep-lat] for more details

RELATIVISTIC BOSE GAS

FREE LANGEVIN DYNAMICS IN MOMENTUM SPACE

$$\frac{\partial}{\partial\theta}\phi^{\mathrm{R}}_{a,p}(\theta) = K^{\mathrm{R}}_{a,p}(\theta) + \eta_{a,p}(\theta)$$
$$\frac{\partial}{\partial\theta}\phi^{\mathrm{I}}_{a,p}(\theta) = K^{\mathrm{I}}_{a,p}(\theta)$$

$$K^{\mathrm{R}}_{a,p} = -A_p\phi^{\mathrm{R}}_{a,p} + iB_p\varepsilon_{ab}\phi^{\mathrm{I}}_{b,p}$$
$$K^{\mathrm{I}}_{a,p} = -A_p\phi^{\mathrm{I}}_{a,p} - iB_p\varepsilon_{ab}\phi^{\mathrm{R}}_{b,p}$$

$$A_p = m^2 + 4\sum_{i=1}^{3}\sin^2\frac{p_i}{2} + 2\left(1 - \cosh\mu\cos p_4\right)$$
$$B_p = 2\sinh\mu\sin p_4$$

Relativistic Bose gas

free Langevin dynamics

- solution:

$$\phi_a^{\rm R}(\theta,p) = e^{-A_p\theta}\left[\cos(B_p\theta)\phi_a^{\rm R}(0,p) + i\sin(B_p\theta)\epsilon_{ab}\phi_b^{\rm I}(0,p)\right] + \int_0^\theta ds\, e^{-A_p(\theta-s)}\cos[B_p(\theta-s)]\eta_a(s,p)$$

$$\phi_a^{\rm I}(\theta,p) = e^{-A_p\theta}\left[\cos(B_p\theta)\phi_a^{\rm I}(0,p) - i\sin(B_p\theta)\epsilon_{ab}\phi_b^{\rm R}(0,p)\right] - i\int_0^\theta ds\, e^{-A_p(\theta-s)}\sin[B_p(\theta-s)]\epsilon_{ab}\eta_b(s,p)$$

- convergence provided $A_p > 0 \Rightarrow 4\sinh^2\frac{\mu}{2} < m^2$
- standard (in)stability for free Bose gas

RELATIVISTIC BOSE GAS

FREE LANGEVIN DYNAMICS

- convergence of two-point functions (provided $A_p > 0$)

$$\lim_{\theta\to\infty} \langle \phi^{\mathrm{R}}_{a,-p}(\theta)\phi^{\mathrm{R}}_{b,p'}(\theta)\rangle = \delta_{ab}\delta_{pp'}\frac{1}{2A_p}\frac{2A_p^2+B_p^2}{A_p^2+B_p^2}$$

$$\lim_{\theta\to\infty} \langle \phi^{\mathrm{I}}_{a,-p}(\theta)\phi^{\mathrm{I}}_{b,p'}(\theta)\rangle = \delta_{ab}\delta_{pp'}\frac{1}{2A_p}\frac{B_p^2}{A_p^2+B_p^2}$$

$$\lim_{\theta\to\infty} \langle \phi^{\mathrm{R}}_{a,-p}(\theta)\phi^{\mathrm{I}}_{b,p'}(\theta)\rangle = \varepsilon_{ab}\delta_{pp'}\frac{i}{2}\frac{B_p}{A_p^2+B_p^2}$$

- structure agrees with symmetry of Langevin dynamics
- observables constructed with these two-point functions agree with standard expressions

RELATIVISTIC BOSE GAS

FREE LANGEVIN DYNAMICS

- discretized Langevin equations

$$\phi(n+1) = [1+\epsilon M]\phi(n) + \sqrt{\epsilon}\eta(n)$$

  are stable provided that $|1+\epsilon M| < 1$
- in this theory: $A_p - \frac{\epsilon}{2}\left(A_p^2 + B_p^2\right) > 0$
- constraint from high momentum modes

$$\epsilon < \frac{2}{4d + m^2 + 2(\cosh\mu - 1)}$$

- $\mu < m$: modest bound on ϵ
- $\mu \gg m$: eventually $\epsilon < e^{-\mu}$ (however, in region where lattice artefacts are severe)

FOKKER-PLANCK EQUATIONS

REAL AND COMPLEX DISTRIBUTIONS

- complex distribution

$$\langle O[\phi,\theta]\rangle_\eta = \int D\phi\, P[\phi,\theta] O[\phi]$$

- satisfies the Fokker-Planck equation

$$\frac{\partial P[\phi,\theta]}{\partial\theta} = \sum_x \frac{\delta}{\delta\phi_{a,x}(\theta)}\left(\frac{\delta}{\delta\phi_{a,x}(\theta)} + \frac{\delta S[\phi]}{\delta\phi_{a,x}(\theta)}\right) P[\phi,\theta]$$

- stationary solution $P[\phi] \sim e^{-S[\phi]}$
- not appropriate for the real Langevin process

FOKKER-PLANCK EQUATIONS

REAL AND COMPLEX DISTRIBUTIONS

- real distribution

$$\langle O[\phi,\theta]\rangle_\eta = \int D\phi^{\mathrm{R}} D\phi^{\mathrm{I}}\, \rho[\phi^{\mathrm{R}},\phi^{\mathrm{I}},\theta] O[\phi^{\mathrm{R}} + i\phi^{\mathrm{I}}]$$

- satisfies the extended Fokker-Planck equation

$$\frac{\partial \rho[\phi^{\mathrm{R}},\phi^{\mathrm{I}},\theta]}{\partial\theta} = \sum_x \left[\frac{\delta}{\delta\phi^{\mathrm{R}}_{a,x}(\theta)} \left(\frac{\delta}{\delta\phi^{\mathrm{R}}_{a,x}(\theta)} - K^{\mathrm{R}}_{a,x}(\theta)\right) - \frac{\delta}{\delta\phi^{\mathrm{I}}_{a,x}(\theta)} K^{\mathrm{I}}_{a,x}(\theta) \right] \rho[\phi^{\mathrm{R}},\phi^{\mathrm{I}},\theta]$$

- stationary solutions not known in general!

FOKKER-PLANCK EQUATION

REAL DISTRIBUTION

- look for stationary solution ignoring interactions

$$\sum_p \Bigg[\frac{\delta}{\delta\phi^{\mathrm{R}}_{a,p}} \frac{\delta}{\delta\phi^{\mathrm{R}}_{a,-p}} + \left(A_p\phi^{\mathrm{R}}_{a,p} - iB_p\varepsilon_{ab}\phi^{\mathrm{I}}_{b,p}\right)\frac{\delta}{\delta\phi^{\mathrm{R}}_{a,p}}$$

$$+ \left(A_p\phi^{\mathrm{I}}_{a,p} + iB_p\varepsilon_{ab}\phi^{\mathrm{R}}_{b,p}\right)\frac{\delta}{\delta\phi^{\mathrm{I}}_{a,p}} + 2A_p\Bigg]\rho[\phi^{\mathrm{R}}, \phi^{\mathrm{I}}] = 0$$

- Gaussian problem

FOKKER-PLANCK EQUATION

REAL DISTRIBUTION

- solution

$$\rho[\phi^{\mathrm{R}}, \phi^{\mathrm{I}}] = N \exp\left[-\sum_p \left(\alpha_p \phi^{\mathrm{R}}_{a,-p}\phi^{\mathrm{R}}_{a,p} + \beta_p \phi^{\mathrm{I}}_{a,-p}\phi^{\mathrm{I}}_{a,p} + 2i\varepsilon_{ab}\gamma_p \phi^{\mathrm{R}}_{a,-p}\phi^{\mathrm{I}}_{b,p}\right)\right]$$

$$\alpha_p = A_p \qquad \beta_p = \frac{A_p}{B_p^2}\left(2A_p^2 + B_p^2\right) \qquad \gamma_p = \frac{A_p^2}{B_p}$$

- generalized partition function

$$Z = \prod_p \int d\phi^{\mathrm{R}}_p d\phi^{\mathrm{I}}_p \, \rho[\phi^{\mathrm{R}}, \phi^{\mathrm{I}}] = \mathcal{N} \prod_p \frac{1}{\alpha_p\beta_p - \gamma_p^2}$$

- Gaussian integrals converge provided $A_p > 0$

FOKKER-PLANCK EQUATION

REAL DISTRIBUTION

- correlation functions

$$\langle \phi^{\mathrm{R}}_{a,-p} \phi^{\mathrm{R}}_{a,p} \rangle = -\frac{\partial \ln Z}{\partial \alpha_p} = \frac{\beta_p}{\alpha_p \beta_p - \gamma_p^2} = \frac{1}{A_p} \frac{2A_p^2 + B_p^2}{A_p^2 + B_p^2}$$

$$\langle \phi^{\mathrm{I}}_{a,-p} \phi^{\mathrm{I}}_{a,p} \rangle = -\frac{\partial \ln Z}{\partial \beta_p} = \frac{\alpha_p}{\alpha_p \beta_p - \gamma_p^2} = \frac{1}{A_p} \frac{B_p^2}{A_p^2 + B_p^2}$$

$$2i\varepsilon_{ab} \langle \phi^{\mathrm{R}}_{a,-p} \phi^{\mathrm{I}}_{b,p} \rangle = -\frac{\partial \ln Z}{\partial \gamma_p} = \frac{-2\gamma_p}{\alpha_p \beta_p - \gamma_p^2} = \frac{-2B_p}{A_p^2 + B_p^2}$$

- agree with solution of Langevin process when $\theta \to \infty$

FOKKER-PLANCK EQUATION

REAL DISTRIBUTION

- distribution is singular as $p_4 \to 0$ or $\mu \to 0$

why?

- mode with $p_4 = 0$ is purely real, not complexified
- when $\mu = 0$, no need for complexification

$$\rho[\phi^{\mathrm{R}}, \phi^{\mathrm{I}}] = P[\phi^{\mathrm{R}}]\delta(\phi^{\mathrm{I}})$$

- include interactions?

MEAN FIELD APPROXIMATION

LANGEVIN DYNAMICS

- include interactions on the mean field level
- Langevin equations contain terms of form $\lambda\phi^3$
- Gaussian factorization: example

$$\phi^{\rm R}_{b,x}\phi^{\rm I}_{b,x}\phi^{\rm R}_{a,x} \to \langle\phi^{\rm R}_{b,x}\phi^{\rm I}_{b,x}\rangle\phi^{\rm R}_{a,x} + \langle\phi^{\rm R}_{b,x}\phi^{\rm R}_{a,x}\rangle\phi^{\rm I}_{b,x} + \langle\phi^{\rm I}_{b,x}\phi^{\rm R}_{a,x}\rangle\phi^{\rm R}_{b,x}$$

- solve for fixed points, etc
- when all the dust settles:

$$A_p \to \mathcal{A}_p = A_p + 4\lambda\langle|\phi|^2\rangle$$

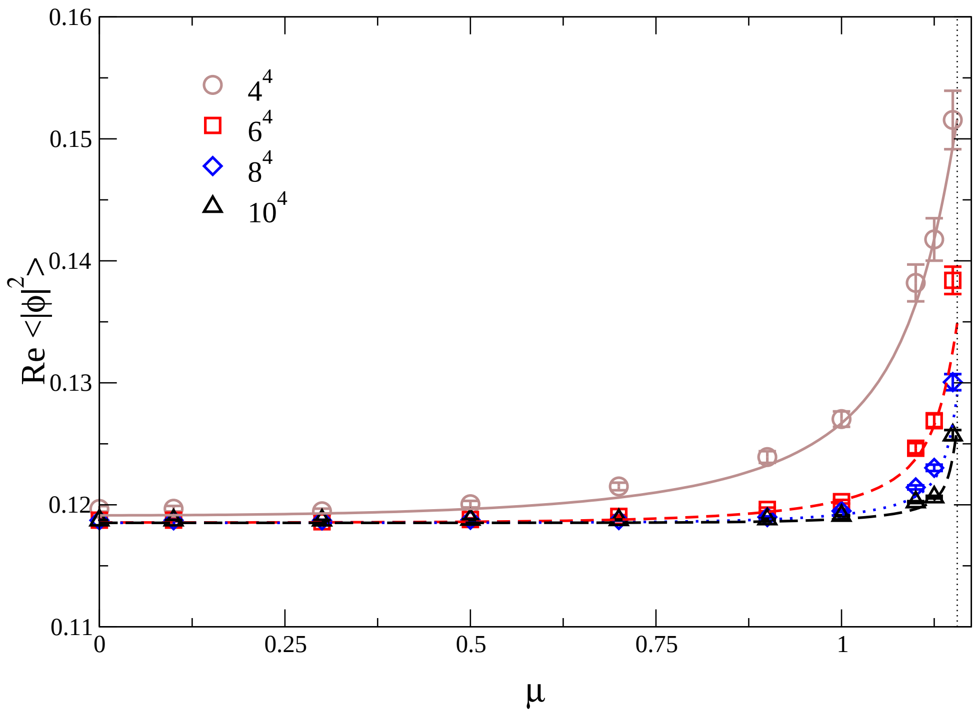
as expected (mean field mass/tadpole resummation)

- solve mass from self-consistent gap equation

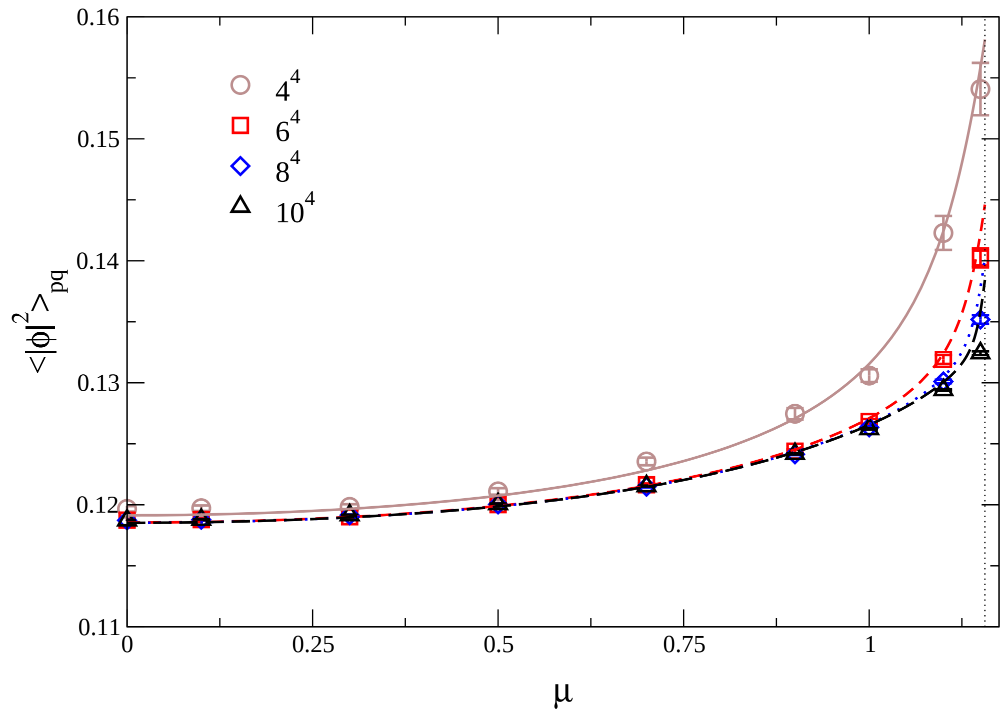
Relativistic Bose gas

MEAN FIELD COMPARISON

$$\langle|\phi|^2\rangle$$



complex



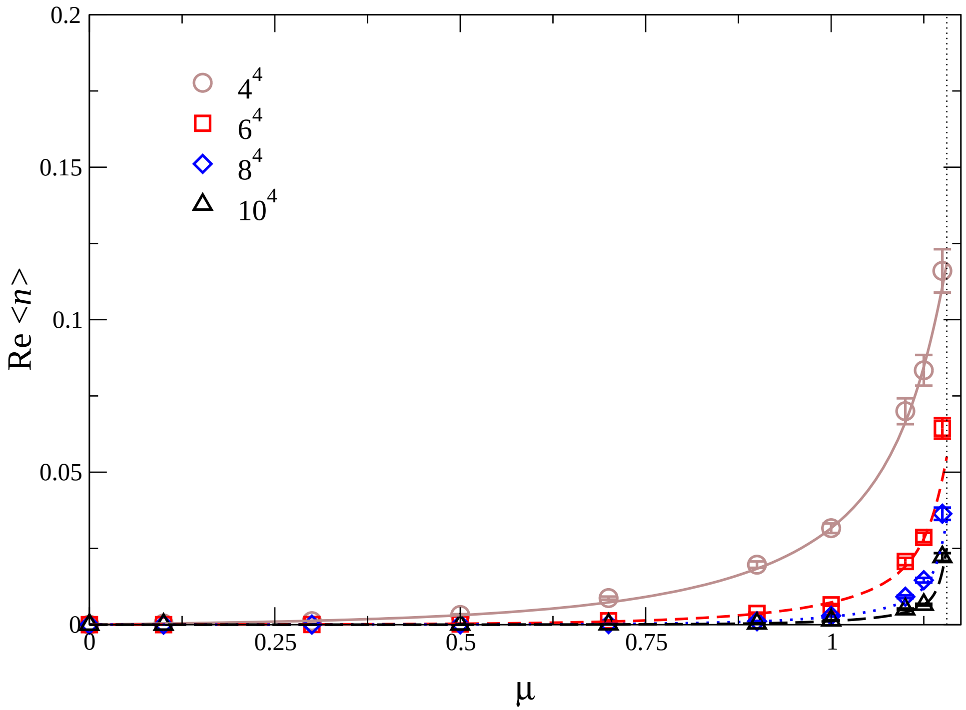
phase quenched

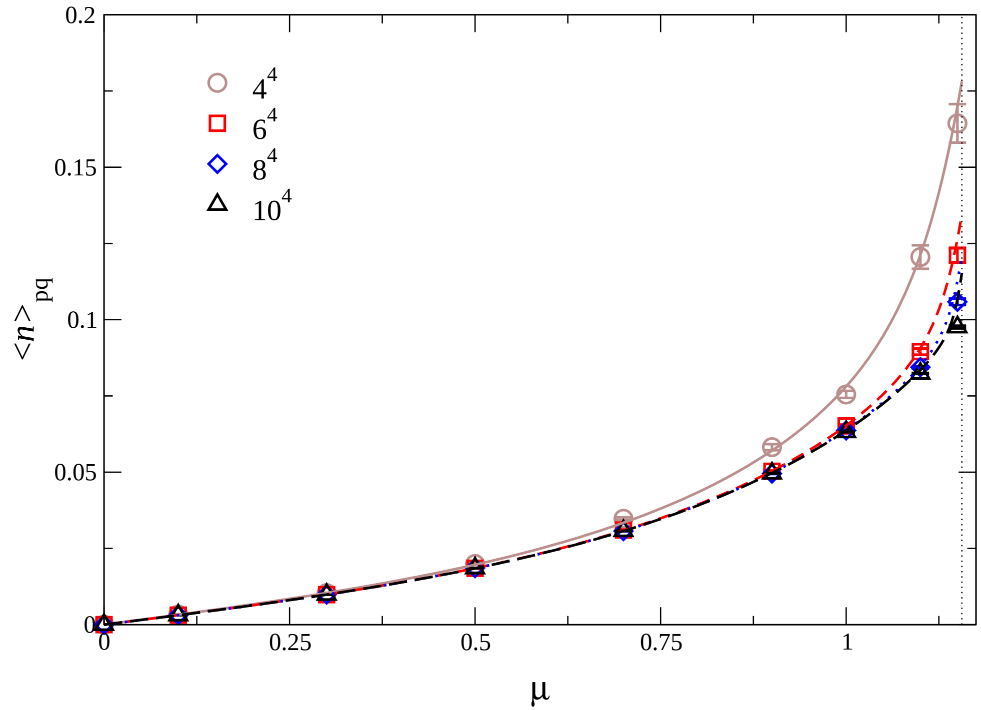
lines are mean field predictions

Relativistic Bose gas

MEAN FIELD COMPARISON

density





complex

phase quenched

lines are mean field predictions

RELATIVISTIC BOSE GAS

MEAN FIELD ANALYSIS

- mean field analysis in noncondensed phase (Silver Blaze region)
- can be analyzed in detail
- agreement with numerical results

- extension to condensed phase
- include $\langle \phi^{\rm R}_{a,x} \rangle \neq 0$
- in progress....

Summary & outlook

many stimulating results

- one link models: excellent agreement
- relativistic Bose gas: phase transition and Silver Blaze
- QCD with static quarks: encouraging

why does it work?
partly understood in simple models and relativistic Bose gas

in progress:

- more analytical insight in the relativistic Bose gas
- QCD with static and dynamical quarks
- ...