RENORMALIZABILITY OF HOT CLASSICAL FIELD THEORY

GERT AARTS
Institute for Theoretical Physics, Utrecht University
Princetonplein 5, 3508 TA Utrecht, the Netherlands

I discuss the possibility of using classical field theory to approximate hot, real-time quantum field theory. I calculate, in a scalar theory, the classical two point and four point function in perturbation theory. The counterterms needed to make the classical correlation functions finite are dictated by the superrenormalizability of the static theory. The classical expressions approximate the quantum ones, when the classical parameters are chosen according to the dimensional reduction matching rules. I end with an outlook to gauge theories.

1 Introduction

The classical approximation was suggested several years ago [1] to make non-perturbative calculations of real-time correlation functions at finite temperature possible, namely by the use of numerical simulations [2]. In this talk I discuss whether classical field theory can be used to approximate the quantum theory, by doing a perturbative calculation in $\lambda\phi^4$ theory. Almost everything I say can be found in ref [3], written together with Jan Smit.

2 Dimensional reduction for static quantities

I start with a brief introduction to the dimensional reduction approach for time independent quantities [4]. The question is how to calculate (in a reliable way) static equilibrium quantities, such as the critical temperature and the order of the phase transition, in a weakly coupled quantum field theory at finite temperature T. In the Matsubara formalism, the field $\phi(\mathbf{x},\tau)$ obeys periodic boundary conditions in imaginary time, $\phi(\mathbf{x},0) = \phi(\mathbf{x},\beta)$, and hence can be decomposed as

$$\phi(\mathbf{x},\tau) = T\sum_n e^{i\omega_n\tau}\phi_n(\mathbf{x}),$$

with the Matsubara frequencies $\omega_n = 2\pi nT$. The dimensional reduction approach is the following: an effective 3-d theory is constructed for $\phi_0(\mathbf{x})$, which can be used e.g. non-perturbatively. This effective theory is chosen to have the same form as the ϕ_0 part of the original theory, but with effective parameters. The parameters in the 3-d theory, $m^2_{\rm eff}$ and $\lambda_{\rm eff}$, are determined by perturbatively matching 3-d correlation functions to the full 4-d correlation functions. A nice property of the effective theory is that it is superrenormalizable. Hence, the coupling constant only receives a finite correction due to matching, $\lambda_{\text{eff}} \approx \lambda$. The effective mass parameter is written as $m^2_{\text{eff}} = m^2 - \delta m^2$. δm^2 takes care of the divergences in the 3-d theory, namely in the one loop tadpole and the two loop setting sun diagram. The finite part m^2 contains in particular the thermal mass, $m^2 \approx \lambda T^2/24\hbar$, where I indicated explicitly the $\hbar$ dependence.

3 What about time dependent correlation functions?

The question is now if this approach can be extended to calculate also real-time quantities, such as transition rates, the plasmon frequency and damping rates.

Consider the partition function for the effective theory ($\phi = \phi_0$)

$$Z_{\text{3-d}} = \int \mathcal{D}\phi\, e^{-\beta V(\phi)}, \qquad V = \int d^3x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2_{\text{eff}}\phi^2 + \frac{\lambda_{\text{eff}}}{4!}\phi^4\right).$$

The form is that of the classical partition function,

$$Z_{\text{cl}} = \int \mathcal{D}\pi\mathcal{D}\phi\, e^{-\beta H(\pi,\phi)}, \qquad H = \int d^3x\, \frac{1}{2}\pi^2 + V(\phi),$$

after the integration over the momenta has been performed. Therefore, let's take a look at time dependent *classical* correlation functions. The classical two point function is given by ($x = (\mathbf{x}, t)$)

$$S(x - x') = \langle\phi(x)\phi(x')\rangle_{\text{cl}} = Z^{-1}_{\text{cl}} \int \mathcal{D}\pi\mathcal{D}\phi\, e^{-\beta H(\pi,\phi)}\, \phi(x)\phi(x'), \tag{1}$$

with $\phi(x)$ the solution of the classical equations of motion $\dot{\phi}(x) = \{\phi(x), H\}$, $\dot{\pi}(x) = \{\pi(x), H\}$, with the initial conditions $\phi(\mathbf{x}, t_0) = \phi(\mathbf{x})$, $\pi(\mathbf{x}, t_0) = \pi(\mathbf{x})$. The integration over phase space is over the initial conditions at $t = t_0$, weighted with the Boltzmann weight.

This definition (1) raises two obvious questions

- What about the divergences? When $t = t'$, the integration over the momenta is trivial, and we are back in the 3-d superrenormalizable theory, what happens to the 3-d divergences when $t \neq t'$?

- What about the relation with time dependent correlation functions in the full quantum field theory? Again, when $t = t'$, the matching relations make sure that the 3-d correlation functions approximate the full 4-d ones, what happens when $t \neq t'$?

4 Perturbative expansion in the classical field theory

To answer the above stated questions, I do a perturbative calculation of the two point and the four point function to order $\lambda_{\rm eff}^2$. The perturbative solution of the equations of motion takes the form

$$
\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, [\phi(\mathbf{k})\cos\omega_{\mathbf{k}}(t-t_0) + \frac{\pi(\mathbf{k})}{\omega_{\mathbf{k}}}\sin\omega_{\mathbf{k}}(t-t_0)] \\
&\quad -\lambda_{\rm eff}\int d^4x'\, G_0^R(x-x')\frac{1}{3!}\phi_0^3(x') + \ldots
\end{aligned}
$$

Here $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2$, and the retarded Green function is defined by

$$
G_{\rm cl}^R(x-x') = -\theta(t-t')\langle\{\phi(x),\phi(x')\}\rangle_{\rm cl}. \tag{2}
$$

The free two point and retarded Green function are given by

$$
S_0(\mathbf{k},t) = T\frac{\cos\omega_{\mathbf{k}}t}{\omega_{\mathbf{k}}^2}, \qquad G_0^R(\mathbf{k},t) = \theta(t)\frac{\sin\omega_{\mathbf{k}}t}{\omega_{\mathbf{k}}}. \tag{3a,b}
$$

It is now straightforward to calculate the two point function, defined in (1). This gives the connected two point function, but it is easy to identify the classical 1PI diagrams and in particular the retarded self energy. It is convenient to work in (temporal) momentum space, just as in thermal field theory.

Let me make a small sidestep here to show the relation with the quantum theory. The quantum retarded Green function is defined by

$$
G_{\rm qm}^R(x-x') = i\theta(t-t')\langle[\phi(x),\phi(x')]\rangle_{\rm qm}.
$$

The free retarded Green function is given by (3b), the classical expression. It is independent of T and $\hbar$. The quantum analogon of the two point function is

$$
F(x-x') = \frac{1}{2}\langle\phi(x)\phi(x') + \phi(x')\phi(x)\rangle_{\rm qm}.
$$

In the free case it is

$$
F_0(\mathbf{k},t) = [n(\omega_{\mathbf{k}}) + \frac{1}{2}]\frac{\cos\omega_{\mathbf{k}}t}{\omega_{\mathbf{k}}} = T\sum_n \frac{\cos\omega_{\mathbf{k}}t}{\omega_n^2+\omega_{\mathbf{k}}^2}, \quad n(\omega) = \left(e^{\beta\omega}-1\right)^{-1}.
$$

The classical two point function (3a) is just the $n=0$ term, similar as in the dimensional reduction approach in imaginary time. The relation between diagrams in the classical and the quantum theory can be completely understood by a suitable formulation of the real-time formalism of thermal field theory.

5 Classical retarded self energy and vertex function

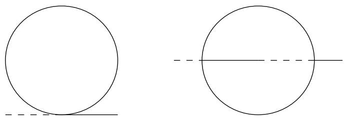

Figure 1: Retarded self energy: one loop and setting sun diagram. The full line represents S_0 and the dashed-full line G_0^R.

The one loop classical retarded self energy is given by the tadpole diagram in fig 1. It is independent of the external momenta, and hence specific problems related to time dependence play no role. The diagram is linear divergent, and the divergence is canceled with δm^2. The result is the same as in the dimensional reduction approach.

The two loop setting sun diagram is more interesting. It has a complicated momentum dependence, a logarithmic divergence when $t = t'$, and it contains an imaginary part, which gives e.g. Landau damping. The diagram (see fig 1) is

$$\begin{aligned} \Sigma^{\rm sun}_{R,\rm cl}(p) &= -\frac{1}{2}\lambda^2_{\rm eff}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} S_0(k_1)S_0(k_2)G_0^R(p-k_1-k_2) \\ &= -\frac{\lambda^2_{\rm eff}T^2}{6}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{\omega^2_{\mathbf{k}_1}\omega^2_{\mathbf{k}_2}\omega^2_{\mathbf{k}_3}} + \frac{p^0}{T}\int\frac{d\Omega}{2\pi}\frac{w(\mathbf{p},\Omega)}{p^0+i\epsilon+\Omega}, \end{aligned} \tag{4}$$

with $\omega_{\mathbf{k}_3} = \omega_{\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2}$ and the typical scattering integral

$$w(\mathbf{p},\Omega) = \frac{\lambda^2_{\rm eff}}{6}\sum_{\{\pm\}}\int\prod_{j=1}^{3}\left[\frac{d^3k_j}{(2\pi)^3 2\omega_{\mathbf{k}_j}}\frac{T}{\omega_{\mathbf{k}_j}}\right](2\pi)^4\delta(\Omega\pm\omega_{\mathbf{k}_1}\pm\omega_{\mathbf{k}_2}\pm\omega_{\mathbf{k}_3}).$$

$T/\omega_{\mathbf{k}}$ is the classical thermal distribution function, and the sum is over all $+$'s and $-$'s. The first term in (4) is independent of p^0, real and is the standard dimensional reduction setting sun result. Its logarithmic divergence is canceled by δm^2. The second term in (4) contains all the p^0 dependence, it is finite, it has an imaginary part and it represents the corresponding leading order quantum expression for large T and small $\lambda_{\rm eff}$. For example, the classical plasmon damping rate is

$$\gamma_{\rm cl} = -\frac{{\rm Im}\,\Sigma^{\rm sun}_{R,\rm cl}(\mathbf{0},m)}{2m} = \frac{\lambda^2_{\rm eff}T^2}{1536\pi m} \approx \frac{\lambda\sqrt{\lambda\hbar}T}{128\sqrt{6}\pi}, \tag{5}$$

where I used the matching results from section 2. Note that $\hbar$ only enters through m. (5) is indeed the leading order quantum result

$$\gamma_{\rm qm} = \frac{\lambda\sqrt{\lambda\hbar}T}{128\sqrt{6}\pi}\left(1+\mathcal{O}(\sqrt{\hbar\lambda}\log\hbar\lambda)\right).$$

Finally, the classical four point function is finite, as in the static superrenormalizable theory, and it reproduces the leading order quantum vertex function.

6 Conclusion and outlook to gauge theories

To summarize, I have considered $\lambda\phi^4$ theory (to two loops), with the conclusion that the time dependent classical theory is renormalizable, i.e. local counterterms are sufficient to make the correlation functions finite. Furthermore the classical expressions approximate the quantum ones. Hard thermal loop effects (which is simply the thermal mass) are easily incorporated because of momentum independence. I stress that the strategy followed is not that of integrating out high momentum modes, which leads to an effective theory with an intermediate cutoff [5].

Is there any use for gauge theories? Here the hard thermal loops effects are much more complicated because of momentum dependence. The one loop classical self energy is linear divergent. The question is if this divergence can be canceled with a (non-local) counterterm. A way to do this might be to use the effective equations of motion, derived within a kinetic theory framework [6], namely

$$D_\mu F^{\mu\nu} = j^\nu, \qquad v^\mu D_\mu w(x,\mathbf{v}) = \mathbf{v}\cdot\mathbf{E}(x),$$
$$j^\mu(x) = 2m^2\int\frac{d\mathbf{v}}{4\pi}v^\mu w(x,\mathbf{v}), \qquad m^2 = \frac{1}{6}g^2NT^2 - \frac{1}{\pi^2}g^2NT\Lambda,$$

with $v^\mu = (1,\mathbf{v})$, $\mathbf{v}\cdot\mathbf{v} = 1$. The mass parameter now also contains a momentum cutoff (Λ) dependent counterterm, just as in the scalar theory. To preserve gauge invariance, it should be formulated on a lattice.

Acknowledgments

This work is supported by FOM.

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