NONEQUILIBRIUM FIELDS: EXACT AND TRUNCATED DYNAMICS[a]

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The nonperturbative real-time evolution of quantum fields out of equilibrium is often solved using a mean-field or Hartree approximation or by applying effective action methods. In order to investigate the validity of these truncations, we implement similar methods in *classical* scalar field theory and compare the approximate dynamics with the full nonlinear evolution. Numerical results are shown for the early-time behaviour, the role of approximate fixed points, and thermalization.

1 Testing truncations?

An understanding of the nonperturbative evolution of quantum fields away from equilibrium is needed in many physical situations. Canonical examples are the universe at the end of inflation and the early stages of a heavy-ion collision. In both cases one would like to calculate how the energy initially contained in the inflaton or heavy ions is redistributed over the available degrees of freedom, leading to a hot universe and a thermal quark-gluon plasma respectively.

Because of the inherent real-time nature and the nonequilibrium setting, standard euclidean lattice methods are not applicable and approximations have to be introduced. In the last ten years or so a lot of attention has been given to mean-field approximations, in which the dynamics of a mean field coupled to Gaussian (quadratic) fluctuations is solved self-consistently. Well-known examples are the Hartree and leading-order large N_f approximations, with N_f the number of scalar or fermion fields. Several paths going beyond homogeneous mean fields have been explored recently as well. For a list of references, see [1]. It goes without saying that the use of approximations will introduce deviations from the true evolution in quantum field theory. In order to gain trust in a particular method, it would be helpful to compare the approximate dynamics with the exact one. Unfortunately, in QFT such tests are not easy since the exact nonperturbative evolution is not known. Therefore we decided to use classical fields, regularized on a lattice, instead. In many aspects classical fields are similar to quantum fields, e.g. scattering is

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present and the role of the thermodynamic limit at fixed lattice spacing can be investigated. Furthermore, most approximation methods can be implemented as well, as they do not involve $\hbar$ directly. Last but not least, the 'exact' evolution can be calculated numerically which allows for a direct comparison.

2 Classical Hartree and beyond

We consider classical ϕ^4 theory in $1+1$ dimensions. The dynamics is determined by the equation of motion $\partial_t^2\phi(x,t) = (\partial_x^2 - m^2)\phi(x,t) - \lambda\phi^3(x,t)/2$, given initial conditions $\phi(x,0)$ and $\pi(x,0) = \partial_t\phi(x,0)$ that are taken from a probability distribution $\rho[\pi,\phi]$. Ensemble averages are denoted with $\langle\cdot\rangle$. The exact nonlinear evolution can be calculated by sampling initial conditions from ρ and solving the e.o.m. numerically for each of them.

A classical Hartree-type of approximation can be implemented by replacing $\lambda\phi^3$ with $3\lambda\phi\langle\phi^2\rangle$ in the equation above. For translationally invariant systems $\langle\phi^2\rangle \equiv \langle\phi^2(x,t)\rangle$ is independent of x and the Hartree e.o.m. can be written conveniently in momentum space as $\partial_t^2\phi(q,t) = -\bar{\omega}_q^2\phi(q,t)$. The effective frequency squared is $\bar{\omega}_q^2 = \omega_q^2 + \frac{3}{2}\lambda\langle\phi^2\rangle$, with $\omega_q^2 = q^2 + m^2$. The unequal-time two-point function $S(x-y;t,t') = \langle\phi(x,t)\phi(y,t')\rangle$ obeys in this approximation the usual mean-field equation of motion $(\partial_t^2 + \bar{\omega}_q^2)S(q;t,t') = 0$. For the Hartree approximation with inhomogeneous mean fields in the quantized version of this model, see [2].

Beyond the Hartree approximation unequal-time formulations typically become nonlocal in time. This is a disadvantage when numerical solutions are required. Therefore we reformulate the Hartree dynamics in terms of three basic equal-time two-point functions: $G_{\psi\psi'}(x-y,t) = \langle\psi(x,t)\psi'(y,t)\rangle$, with $\psi = \{\pi,\phi\}$. In terms of these the Hartree evolution equations read

$$\begin{aligned}
\partial_t G_{\phi\phi}(q,t) &= 2G_{\pi\phi}(q,t),\\
\partial_t G_{\pi\phi}(q,t) &= -\bar{\omega}_q^2 G_{\phi\phi}(q,t) + G_{\pi\pi}(q,t),\\
\partial_t G_{\pi\pi}(q,t) &= -2\bar{\omega}_q^2 G_{\pi\phi}(q,t).
\end{aligned} \tag{1}$$

These equations conserve $\alpha^{-2}(q) = G_{\phi\phi}(q,t)G_{\pi\pi}(q,t) - G_{\pi\phi}^2(q,t)$ for each q.

The evolution equations beyond Hartree include dynamical equal-time four-point functions. In the case of an $O(N_f)$ theory they contain all $1/N_f$ corrections. Details can be found in [1]. It remains to specify the initial probability distribution. We take a Gaussian ensemble with $\langle\psi(q,0)\psi'(q',0)\rangle = G_{\psi\psi'}(q,0)2\pi\delta(q+q')\delta_{\psi\psi'}$, and

$$G_{\phi\phi}(q,0) = T_0/(q^2+m^2), \qquad G_{\pi\pi}(q,0) = T_0. \tag{2}$$

Note that this corresponds to the equilibrium ensemble when $\lambda = 0$.

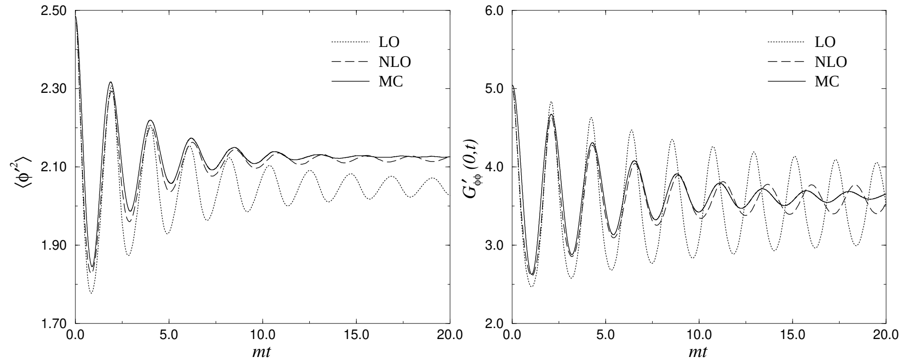

Figure 1: Early-time evolution in the Hartree approximation (LO), with the inclusion of four-point functions (NLO), and 'exact' (MC). Left: mean field squared $\langle\phi'^2\rangle$. Right: zero-mode $G'_{\phi\phi}(q=0,t)$. The initial temperature of the Gaussian ensemble is $T_0'=5$.

3 Early time, fixed points, and thermalization

Typical behaviour at early times is shown in Fig. 1 for two observables. We see that the LO evolution is in qualitative agreement with MC result. At NLO the agreement is impressive. For numerical details I refer again to [1]. Note that primes indicate dimensionless quantities and $T_0' \equiv 3\lambda T_0/m^3$.

An important issue in nonequilibrium field theory is the approach to equilibrium. One of the crucial tests for a truncated evolution is whether it is able to describe the thermalization regime or if the approximation breaks down before. In a classical field theory, a convenient two-point function to investigate is $G_{\pi\pi}(q,t) \equiv T(q,t)$. Using the equilibrium partition function as a guide, we see that this correlator can be interpreted as the effective temperature for a momentum mode q. In equilibrium, $T(q)=T$ for all q. In Fig. 2 (left) we show the time evolution of $T(q,t)$ for three momentum modes (NLO, MC only). It is clear that the modes have a different effective temperature. The system seems to be quasi-stationary but is not thermal.

A remarkable feature is that this behaviour can be understood completely in terms of fixed points in the truncated evolution equations. The stationary points in the Hartree approximation are easily found from Eq. (1), and obey

$$G^*_{\pi\pi}(q) = \bar{\omega}_q^{*2} G^*_{\phi\phi}(q), \qquad G^*_{\pi\phi}(q) = 0, \tag{3}$$

with $\bar{\omega}_q^{*2} = \omega_q^2 + \frac{3}{2}\lambda\langle\phi^2\rangle^*$. These relations can be completed with the expression for the conserved combination at the fixed point

$$\alpha^{-2}(q) = T_0^2/\omega_q^2 = G^*_{\pi\pi}(q) G^*_{\phi\phi}(q). \tag{4}$$

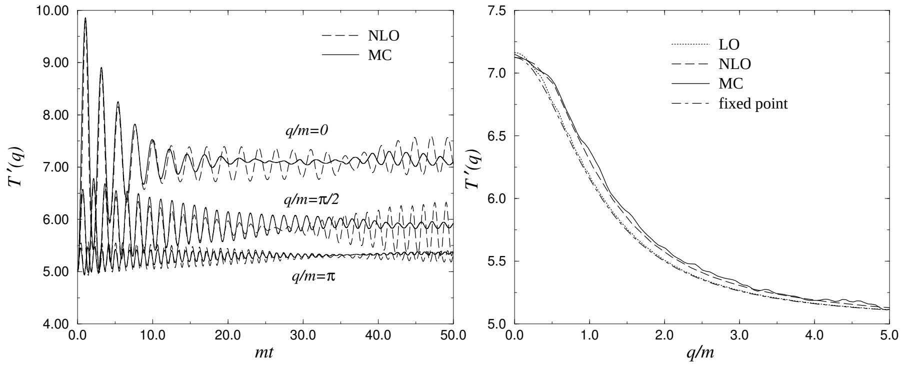

Figure 2: Nonthermal behaviour. Left: effective temperature $T'(q,t)$ for 3 momentum modes. Right: time-averaged temperature profile between $0 < mt < 50$, the fourth line is the analytic expression (6) at the fixed point in the Hartree approximation ($T_0' = 5$).

The first equality follows from the initial ensemble (2). Eqs. (3) and (4) specify the fixed point completely. We find

$$G^*_{\phi\phi}(q) = \frac{T_0}{\bar{\omega}^*_q \omega_q}, \qquad \langle \phi^2 \rangle^* = \int \frac{dq}{2\pi} \frac{T_0}{\bar{\omega}^*_q \omega_q}, \tag{5}$$

(the second expression is a gap equation for $\langle \phi^2 \rangle^*$ and can be written in terms of elliptic functions) and

$$G^*_{\pi\pi}(q) = T_0 \frac{\bar{\omega}^*_q}{\omega_q} = T_0 \left[1 + \frac{3}{2} \frac{\lambda \langle \phi^2 \rangle^*}{\omega^2_q} \right]^{1/2}, \tag{6}$$

i.e., a nonthermal profile of $G^*_{\pi\pi}(q)$ at the fixed point. In order to see whether this fixed-point profile is realized in the dynamical evolution, we show the time average of $G_{\pi\pi}(q,t)$ for the Hartree, NLO, and full evolution in Fig. 2 (right). It is clear that the analytic expression not only describes the Hartree result, but also the profile from the NLO and in particular the full nonlinear evolution surprisingly well. This can be interpreted as a justification for the use of a Hartree approximation in this time interval.

The fate of the fixed point can be determined by being patient. In Fig. 3 (left) we show the evolution of $T(q=0,t)$, for a higher initial temperature T_0 as before. Also shown are the average temperatures $T(t) = N^{-1} \sum_q T(q,t)$. The Hartree evolution remains to be controlled by its fixed point. In the full nonlinear evolution on the other hand, all momentum modes obtain the same temperature. The change from a fixed-point profile to a thermal profile is

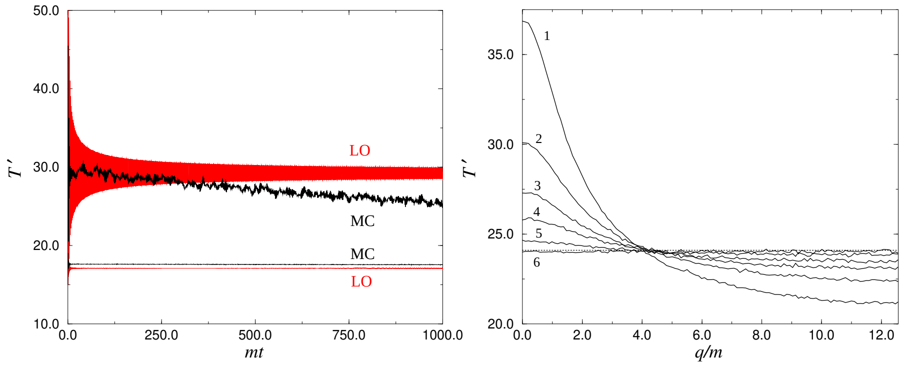

Figure 3: Fate of the fixed point. Left: zero-mode temperature $T'(q=0,t)$ for LO and MC ($T'_0 = 15$). In the Hartree approximation the fixed point governs the dynamics for late times as well, whereas the full nonlinear dynamics shows thermalization. The average temperatures $T'(t)$ appear as straight lines. Right: time-averaged snapshots of the temperature profile (MC only, $T'_0 = 20$, line 1: $0 < mt < 1500, \ldots,$ line 6: $13500 < mt < 15000$). The fixed-point shape slowly disappears and the spectrum becomes thermal (flat).

shown in Fig. 3 (right) for the full evolution only. Due to instabilities, it was not possible to reach such late times using the NLO evolution [1].

4 Summary

In order to gain insight in how well truncated dynamics reproduces the full evolution in nonequilibrium quantum field theory, we studied the corresponding classical problem. We stress that the aim was not to use classical fields to approximate the quantum theory, but rather to study truncated dynamics in a field theory where the exact evolution is available. Our findings are that while at early times the truncated evolution works remarkably well and the outcome can be characterized nicely in terms of fixed points of the associated Hartree equations, it breaks down before the thermalization regime is reached.

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References

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